# Prediction from the One Way Error Components Model with AR(1) Disturbances

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In this paper we extend the works by Baillie and Baltagi [1999] and generalize some results in Baltagi and Li [1992] paper accounting for AR(1) errors in the disturbance term. In particular, we derive six predictors for the one way error components model, as well as their associated asymptotic mean squared error (AMSE) of multi-step prediction in the presence of AR(1) errors in the disturbance term. In addition, we also provide both theoretical and simulation evidence as to the relative efficiency of our alternative predictors. The adequacy of the prediction AMSE formula is also investigated by the use of Monte Carlo methods and indicates that the ordinary optimal predictor performs well for various accuracy criteria.

**Key-words:** Predictors - One way error components model - AR(1) disturbance-Mean squared error (MSE) - Asymptotic mean squared error (AMSE) - Monte Carlo results.

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In this paper we extend the works by Baillie and Baltagi [1999] and generalize some results in Baltagi and Li [1992] paper accounting for AR(1) errors in the disturbance term. In particular, we derive six predictors for the one way error components model, as well as their associated asymptotic mean squared error (AMSE) of multi-step prediction in the presence of AR(1) errors in the disturbance term. In addition, we also provide both theoretical and simulation evidence as to the relative efficiency of our alternative predictors. The adequacy of the prediction AMSE formula is also investigated by the use of Monte Carlo methods and indicates that the ordinary optimal predictor performs well for various accuracy criteria.



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## 1 Introduction

Following the work of Balestra and Nerlove [1966] the regression model with error components, or variance components, has become a popular method for dealing with panel data. A summary of the main features of the model, together with a discussion of some applications, is available in Baltagi [2008], Hsiao [2003] and Maddala [1983 and 1993] among others.

However, relatively little is known about prediction from the model in the one way settings with autoregressive errors in the disturbance term. Assuming that all the regression parameters and the error process parameters are known, the form of the optimal [in the sense of minimum mean squared error (MSE)] predictor has yet to be derived.

This paper extends the works by Baillie and Baltagi [1999] and Baltagi and Li [1992] and investigate some potentially important problems associated with prediction from the one way error component model with AR(1) errors. In particular, we derive various predictors in the presence of AR(1) errors, together with their asymptotic mean squared error (AMSE) of multi-step prediction from the above prediction functions, which accounts for the additional inherent uncertainty associated with estimating both the regression parameters and the error components parameters. The adequacy of the prediction AMSE formula is also investigated by the use of Monte Carlo methods. In addition, we also provide both theoretical and simulation evidence as to the relative efficiency of six alternative predictors; (i) an ordinary predictor, based on the form of the optimal predictor; (ii) a feasible optimal predictor with MLEs replacing population parameters; (iii) a fully feasible optimal predictor with the unknown variance components replaced with consistent estimates [e.g., see Wallace and Hussain, 1969 and Amemiya, 1971]; (iv) a truncated predictor that ignores the error component correction but uses MLEs for its regression parameters; (v) a misspecified predictor which uses OLS estimates of the regression parameters, and (vi) a fixed effects predictor which assumes that the individual and time effects are fixed parameters that can be estimated.

The remainder of the paper proceeds as follows: in Section 2 prediction functions in the presence of AR(1) errors in the disturbance term are derived; asymptotic mean squared error of the derived prediction functions are provided in Section 3. In order to assess the accuracy of the various one way error components model predictors in the presence of AR(1) correlated errors, some Monte Carlo experiments are performed in Section 4 while Section 5 concludes the paper.

## 2 Prediction Functions with AR(1) Errors

Consider the regression model with $T$ pooled time series and $N$ pooled cross-sections defined as,

$$y_{it} = x_{it}'\beta + u_{it}\,, i = 1, 2, \ldots, N \text{ and } t = 1, 2, \ldots, T \tag{1}$$

where $x_{it}$ is a $K$ dimensional vector of explanatory variables associated with the $t$th time period for the $i$th cross section of individuals, firms, or countries. The disturbance $u_{it}$ follows the one way error components model,

$$u_{it} = \mu_i + \nu_{it} \tag{2}$$

where $\mu_i$ denotes the firm-specific effect which is assumed to be $IID\left(0, \sigma_\mu^2\right)$ and $\nu_{it}$ is the remainder disturbance which is also assumed to be $IID\left(0, \sigma_\nu^2\right)$. The $\mu_i$s and $\nu_{it}$s are independent of each other.

In addition, we assume that the remainder error follows a stationary AR(1) process such that

$$\nu_{it} = \rho\nu_{i,t-1} + \varepsilon_{it} \tag{3}$$

with $E(\varepsilon_{it}) = 0$, $Var(\varepsilon_{it}) = \sigma^2_\varepsilon$ and $Var(\nu_{it}) = \sigma^2_\nu = \dfrac{\sigma^2_\varepsilon}{1-\rho^2}$.

With known parameters $\left(\beta, \rho, \sigma^2_\mu, \sigma^2_\nu\right)$ and known future exogenous variables $x_{i,T+h}$, the optimal, i.e., minimum mean square error (MSE) predictor of $y_{i,T+h}$ is denoted by $y_{i,T,h}$ and is the optimal prediction of the $i$th component at time $T$ with $h$ steps ahead forecast horizon.

## 2.1 The Optimal Predictor

The optimal prediction of the $i$th component at time $T$ with $h$ periods ahead denoted $y_{i,T,h}$ is obtained using the methodology developed by Goldberger [1962]. Following Goldberger [1962] and Baltagi [2008] the form of the optimal predictor is given by,

$$y_{i,T,h} = x'_{i,T+h}\beta + w'\Sigma^{-1}u, \quad h \geq 1 \tag{4}$$

where $w = E(u_{i,T+h}u)$, $u$ denotes the disturbances vector and $\Sigma$ its variance-covariance matrix. The expression $w'\Sigma^{-1}u$ is obtained from,

$$\Sigma = \sigma^2_\mu (I_N \otimes \iota_T\iota'_T) + \sigma^2_\nu (I_N \otimes \Gamma)$$

with

$$\Gamma = \begin{pmatrix} 1 & \rho & \cdots & \rho^{T-1} \\ \rho & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \rho \\ \rho^{T-1} & \cdots & \rho & 1 \end{pmatrix} \tag{5}$$

If $C$ is the Prais-Winsten transformation matrix which has the following expression,

$$C = \begin{pmatrix} \sqrt{1-\rho^2} & 0 & 0 & \cdots & 0 \\ -\rho & 1 & 0 & \cdots & 0 \\ 0 & -\rho & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -\rho & 1 \end{pmatrix}$$

we can transform $y$ and get,

$$y^* = (I_N \otimes C)\, y \tag{6}$$

We then get the associated variance covariance matrix

$$\begin{aligned}\Sigma^* &= (I_N \otimes C)\,\Sigma\left(I_N \otimes C^{'}\right) = \sigma_\mu^2\left(I_N \otimes C\iota_T\iota_T^{\prime}C^{'}\right) + \sigma_\varepsilon^2\,(I_N \otimes I_T)\\ &= \sigma_\varepsilon^2\,(I_N \otimes I_T) + \sigma_\mu^2\left(I_N \otimes \iota_T^{\alpha}\iota_T^{\alpha'}\right)\end{aligned}$$

with $\iota_T^{\alpha} = C\iota_T$. It is now easy to derive $\Sigma^{*-1}$ which is,

$$\begin{aligned}\Sigma^{*-1} &= \frac{1}{\sigma_\varepsilon^2} I_N \otimes \left(I_T + \frac{\sigma_\mu^2}{\sigma_\varepsilon^2}\iota_T^{\alpha}\iota_T^{\alpha'}\right)^{-1}\\ &= \frac{1}{\sigma_\varepsilon^2}\left(I_N \otimes I_T - \frac{\sigma_\mu^2}{\sigma_\varepsilon^2 + d\sigma_\mu^2} I_N \otimes \iota_T^{\alpha}\iota_T^{\alpha'}\right) \end{aligned} \tag{7}$$

where

$$d = \iota_T^{\alpha'}\iota_T^{\alpha} = 1 - \rho^2 + (T-1)\,(1-\rho)^2$$

Since

$$\Sigma^* = (I_N \otimes C)\,\Sigma\left(I_N \otimes C^{'}\right)$$

one can deduce that

$$\Sigma^{-1} = \left(I_N \otimes C^{'}\right)\Sigma^{*-1}\,(I_N \otimes C) \tag{8}$$

i.e. the original variance-covariance matrix of the composite error is

$$\Sigma^{-1} = \frac{1}{\sigma_\varepsilon^2}\left(I_N \otimes C^{'}C - \frac{\sigma_\mu^2}{\sigma_\varepsilon^2 + d\sigma_\mu^2} I_N \otimes C^{'}\iota_T^{\alpha}\iota_T^{\alpha'}C\right) \tag{9}$$

Also,

$$\begin{aligned} w^{'} &= E\left(u_{i,T+h}u^{'}\right) = E\left(u_{i,T+h}u_1^{'}, \ldots, u_{i,T+h}u_i^{'}, \ldots, u_{i,T+h}u_T^{'}\right)\\ &= d_i^{'} \otimes \left(\rho^h\sigma_\nu^2\left(\rho^{T-1}, \rho^{T-2}, \cdots 1\right) + \sigma_\mu^2\iota_T^{'}\right)\end{aligned} \tag{10}$$

where

$$d_i^{\prime} = (0, \ldots, 0, \underset{ith}{1}, 0, \ldots, 0)$$

It follows, on the one hand, that

$$\begin{aligned}\left(d_i^{\prime} \otimes \iota_T^{'}\right)\Sigma^{-1} &= \frac{1}{\sigma_\varepsilon^2}\left(d_i^{\prime} \otimes \iota_T^{'}C^{'}C\right) - \frac{\sigma_\mu^2}{\sigma_\varepsilon^2 + d\sigma_\mu^2}\left(d_i^{\prime} \otimes \iota_T^{'}C^{'}\iota_T^{\alpha}\iota_T^{\alpha'}C\right)\\ &= \frac{1}{\sigma_\varepsilon^2 + d\sigma_\mu^2}\left(d_i^{\prime} \otimes \iota_T^{\alpha'}C\right)\end{aligned} \tag{11}$$

On the other hand, one has

$$\left(\rho^{T-1}, \rho^{T-2}, \cdots 1\right) C^{'} C = \left(1-\rho^{2}\right) d_{T}^{'} \tag{12}$$

and

$$\left(\rho^{T-1}, \rho^{T-2}, \cdots 1\right) C^{'} \iota_{T}^{\alpha} = \left(1-\rho^{2}\right) \tag{13}$$

where $d_T'$ is a $T$ vector with 1 as the last element and 0 elsewhere, i.e.

$$d_T' = (0, \ldots, 0, 1)$$

As a consequence,

$$\begin{aligned} w^{'}\Sigma^{-1} &= \frac{\left(1-\rho^{2}\right)\rho^{h}\sigma_{\nu}^{2}}{\sigma_{\varepsilon}^{2}}\left[d_{i}^{'} \otimes d_{T}^{'} - \frac{\sigma_{\mu}^{2}}{\sigma_{\varepsilon}^{2}+d\sigma_{\mu}^{2}}\left(d_{i}^{'} \otimes \iota_{T}^{\alpha'} C\right)\right] + \frac{\sigma_{\mu}^{2}}{\sigma_{\varepsilon}^{2}+d\sigma_{\mu}^{2}}\left(d_{i}^{'} \otimes \iota_{T}^{\alpha'} C\right) \\ &= \rho^{h}\left(d_{i}^{'} \otimes d_{T}^{'}\right) + \frac{\sigma_{\mu}^{2}}{\sigma_{\varepsilon}^{2}+d\sigma_{\mu}^{2}}\left(1-\rho^{h}\right)\left(d_{i}^{'} \otimes \iota_{T}^{\alpha'} C\right) \end{aligned} \tag{14}$$

Set

$$F^{'} = \left(f\, g \ldots g\, h\right) \tag{15}$$

with

$$\begin{cases} f = \eta\left(1-\rho^{h}\right)(1-\rho) \\ g = \eta\left(1-\rho^{h}\right)(1-\rho)^{2} \\ h = \rho^{h} + \eta\left(1-\rho^{h}\right)(1-\rho) \end{cases} \tag{16}$$

and $\eta = \dfrac{\sigma_{\mu}^{2}}{\sigma_{\varepsilon}^{2}+d\sigma_{\mu}^{2}}$.

As a result,

$$w^{'}\Sigma^{-1}u = F^{'}u_{i} \tag{17}$$

leading to the following expression of the predictor,

$$y_{i,T,h} = x_{i,T+h}^{'}\beta + F^{'}u_{i}, \quad h \geq 1\,. \tag{18}$$

It can be shown [See Appendix A] that equation (18) is consistent with the formula of the one way error component model with AR(1) errors in Baltagi and Li [1992] and Baltagi [2008] when $h = 1$. In fact, equation (18) generalizes their findings as a prediction formula for $h$ periods ahead ($h \geq 1$). Contrary to these authors, we rather stress the one-step prediction process instead of emphasizing the Prais-Winsten transformed errors $u_i^* = Cu_i$. It then appears that, in predicting the one-way AR(1) error components model, not only the first individual error receive a special treatment but the last error also has a specific weight $h$ while the other errors have the same weight $g$. In the same logical as Baltagi and Li [1992] and Baltagi [2008], we shall point out that,

(i) if $\sigma_\mu^2 = 0$, (only serial correlation is considered), then $F'u_i$ collapses into $\rho^h u_{iT}$, and

(ii) if $\rho = 0$ (no serial correlation), then $F'u_i$ reduces to $\dfrac{\sigma_\mu^2}{\sigma_\varepsilon^2 + T\sigma_\mu^2}\sum_{t=1}^{T} u_{it}$ and all errors receive the same weight ($f = g = h = \dfrac{\sigma_\mu^2}{\sigma_\varepsilon^2 + T\sigma_\mu^2}$).

## 2.2 The Feasible Optimal Predictor

The feasible optimal predictor $\hat{y}_{i,T,h}$ is obtained by substituting MLE for corresponding unknown population parameters into (18). In the context of the one way error components regression model, for the case where the true variance components are known, Baltagi and Li [1992] have applied Golberger's [1962] result to show that $\hat{y}_{i,T,h}$ is actually the best linear unbiased predictor (BLUP). Under the AR(1) correlation setting, the feasible optimal predictor for the $i$th individual $h$ periods ahead is given as,

$$\hat{y}_{i,T,h} = x'_{i,T+h}\hat{\beta} + F'u_i, \quad h \geq 1 \tag{19}$$

where

$$\hat{\beta} = \left(X'\Sigma^{-1}X\right)^{-1} X'\Sigma^{-1}y$$

Here, $y' = (y_{11}, \ldots, y_{1T}, \ldots, y_{N1}, \ldots, y_{NT})'$ and $X$ is the $NT \times K$ dimensional matrix with the $i$th row being $x'_{it}$ stacked in the same way as defined for $y$.

## 2.3 The Fully Feasible Optimal Predictor

To obtain the fully feasible optimal predictor, it is necessary to replace the unknown variance components with consistent estimates. Wallace and Hussain [1969] and Amemiya [1971] describe two such estimators. When the parameters in this predictor are replaced with their corresponding MLEs, the feasible predictor or 'ordinary predictor', becomes

$$\widehat{\hat{y}}_{i,T,h} = x'_{i,T+h}\hat{\beta} + \hat{F}'\hat{u}_i \tag{20}$$

In this paper, we derive the asymptotic mean squared error of $\widehat{\hat{y}}_{i,T,h}$ and compare its efficiency with the one of three alternative predictors.

## 2.4 The Truncated Predictor

The first of these alternative predictors is the truncated predictor

$$y^*_{i,T,h} = x'_{i,T+h}\hat{\beta} \tag{21}$$

which is based on efficient estimates of the regression parameters, but is suboptimal in the sense that it ignores the contribution of autocorrelation to the predictor. This truncated predictor corresponds to the expected value predictor in the terminology of Goldberger [1962] and Baillie [1980].

## 2.5 The OLS Predictor

Another predictor of considerable practical importance is the predictor based on inefficient OLS estimates of the regression parameters

$$y_{i,T,h}^{**} = x_{i,T+h}' \hat{\beta}_{OLS} \tag{22}$$

where $\hat{\beta}_{OLS}$ is the least squares estimator and clearly ignores the autocorrelated error components phenomenon in both estimation and formulation of the predictor. Equation (22) corresponds to the situation where the investigator is oblivious to the presence of the error components and merely uses standard regression software to calculate prediction and their associated MSE.

## 2.6 The Fixed Effect Predictor

The predictor for the fixed effects model is obtained using the within transformation i.e.,

$$\tilde{y}_{i,T,h} = x_{i,T+h}' \tilde{\beta} + \tilde{\mu}_i \tag{23}$$

with $\tilde{\beta} = \hat{\beta}_{FE} = (X'QX)^{-1} X'Qy$, $Q = I_N \otimes \left( I_T - \frac{1}{T}\iota_T \iota_T' \right)$, $\tilde{\mu}_i = \bar{y}_{i.} - \tilde{\beta}' \bar{x}_{i.}$, and $\bar{x}_{i.} = \frac{1}{T} \sum_{t=1}^{T} x_{it}$, $\bar{y}_{i.} = \frac{1}{T} \sum_{t=1}^{T} y_{it}$.

# 3 Asymptotic Mean Squared Error (AMSE) of Prediction

To compare the relative efficiency of predictors it is necessary to examine their asymptotic mean squared errors. There is a substantial previous literature on the effect of parameter estimation and the prediction of exogenous variables in the context of dynamic econometric models, e.g., Schmidt [1974, 1977], Lahiri [1975], Baillie [1979, 1980], Yamamoto [1979] and Lütkepohl [1988]. The method of analysis in this paper is similar to the above literature, except that the crucial parameters of interest in the predictor arise from error component effects rather than a dynamic model. For example, Baillie [1980] compared the asymptotic efficiencies of the predictors given by (21), (22) and (23) for the time series regression model with ARMA(p, q) disturbances and Yamamoto |1979] provided a related treatment for the pure AR(1) process. In this paper, we examine the one way error component regression model with AR(1) remainder disturbances and compare the relative efficiency of predictors given in equations (20) to (23).

## 3.1 The Optimal Predictor with Known Parameters

The optimal predictor with known parameters $\left(\beta, \rho, \sigma_{\nu}^{2}, \sigma_{\mu}^{2}\right)$, $y_{i,T,h}$ is defined in (18). By definition, its mean squared error is given by

$$AMSE\left(y_{i,T,h}\right) = E\left(y_{i,T+h} - y_{i,T,h}\right)^{2}$$

Since

$$y_{i,T,h} = x_{i,T+h}^{\prime}\beta + F^{\prime}u_{i}, \quad h \geq 1$$

we can write

$$\begin{aligned} y_{i,T+h} - y_{i,T,h} &= \nu_{i,T+h} + \mu_{i} - F^{\prime}u_{i} \\ &= \nu_{i,T+h} - F^{\prime}\nu_{i} + \mu_{i}\left(1 - F^{\prime}\iota_{T}\right) \end{aligned} \tag{24}$$

so that

$$\begin{aligned} E\left(y_{i,T+h} - y_{i,T,h}\right)^{2} &= E\left(\nu_{i,T+h} - F^{\prime}\nu_{i}\right)^{2} + E\left[\mu_{i}\left(1 - F^{\prime}\iota_{T}\right)\right]^{2} \\ &= \sigma_{\nu}^{2}\left(1 - 2\rho^{h}F^{\prime}\Gamma_{T} + F^{\prime}\Gamma F\right) + \sigma_{\mu}^{2}\left(1 - F^{\prime}\iota_{T}\right)^{2} \end{aligned} \tag{25a}$$

with

$$\Gamma = \begin{pmatrix} 1 & \rho & \cdots & \rho^{T-1} \\ \rho & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \rho \\ \rho^{T-1} & \cdots & \rho & 1 \end{pmatrix} \text{ and } \Gamma_{T} = \begin{pmatrix} \rho^{T-1} \\ \vdots \\ \rho \\ 1 \end{pmatrix} \text{(the last column of } \Gamma\text{)}.$$

In details, one gets

$$E\left(y_{i,T+h} - y_{i,T,h}\right)^{2} = \sigma_{\nu}^{2}\left(1 - \rho^{2h} + \frac{d\left(1+\rho\right)}{1-\rho}f^{2}\right) + \left(1 - \rho^{h} - \left(T - \left(T-2\right)\rho\right)f\right)^{2}\sigma_{\mu}^{2} \tag{25b}$$

## 3.2 The Optimal Predictor with Estimated Parameters

The practically feasible version of the optimal predictor in (18) with MLEs of its parameters $\left(\widehat{\beta}, \widehat{\rho}, \widehat{\sigma}_{\nu}^{2}, \widehat{\sigma}_{\mu}^{2}\right)$ replacing population parameters, has a prediction error derived from

$$y_{i,T+h} - \widehat{\widehat{y}}_{i,T,h} = \left(y_{i,T+h} - y_{i,T,h}\right) + \left(y_{i,T,h} - \widehat{\widehat{y}}_{i,T,h}\right)$$

In particular,

$$
\begin{aligned}
\widehat{\widehat{y}}_{i,T,h}-y_{i,T+h} &= x'_{i,T+h}\left(\hat{\beta}-\beta\right)+\hat{F}'\hat{u}_i-F'u_i \\
&= x'_{i,T+h}\left(\hat{\beta}-\beta\right)-F'u_i+\left[F+\left(\hat{F}-F\right)\right]'\left[u_i+\left(\hat{u}_i-u_i\right)\right] \\
&= \left(x'_{i,T+h}-F'x_i\right)\left(\hat{\beta}-\beta\right)+\left(\hat{F}-F\right)'u_i \\
&\quad -\left(\hat{F}-F\right)'x_i\left(\hat{\beta}-\beta\right) \qquad (26)
\end{aligned}
$$

Here, to get the AMSE, we need to obtain the asymptotic variance covariance matrix for the maximum likelihood estimators of $\gamma=\left(\beta,\rho,\sigma_\nu^2,\sigma_\mu^2\right)$. We have,

$$
n^{\frac{1}{2}}\left(\hat{\gamma}-\gamma\right) \xrightarrow{d} N\left(0,J^{-1}\left(\gamma\right)\right)
$$

where $J\left(\gamma\right)$ is the Fisher information matrix and $n=NT$. We assume that $T$ is fixed and $N$ varies. The Fisher information matrix is then given by,

$$
J\left(\gamma\right)=-E\begin{bmatrix}
\frac{\partial^2 l(\cdot)}{\partial\beta\partial\beta'} & \frac{\partial^2 l(\cdot)}{\partial\beta\partial\rho} & \frac{\partial^2 l(\cdot)}{\partial\beta\partial\sigma_\nu^2} & \frac{\partial^2 l(\cdot)}{\partial\beta\partial\sigma_\mu^2} \\
\frac{\partial^2 l(\cdot)}{\partial\beta\partial\rho} & \frac{\partial^2 l(\cdot)}{(\partial\rho)^2} & \frac{\partial^2 l(\cdot)}{\partial\rho\partial\sigma_\nu^2} & \frac{\partial^2 l(\cdot)}{\partial\rho\partial\sigma_\mu^2} \\
\frac{\partial^2 l(\cdot)}{\partial\beta\partial\sigma_\nu^2} & \frac{\partial^2 l(\cdot)}{\partial\rho\partial\sigma_\nu^2} & \frac{\partial^2 l(\cdot)}{\left(\partial\sigma_\nu^2\right)^2} & \frac{\partial^2 l(\cdot)}{\partial\sigma_\nu^2\partial\sigma_\mu^2} \\
\frac{\partial^2 l(\cdot)}{\partial\beta\partial\sigma_\mu^2} & \frac{\partial^2 l(\cdot)}{\partial\rho\partial\sigma_\mu^2} & \frac{\partial^2 l(\cdot)}{\partial\sigma_\nu^2\partial\sigma_\mu^2} & \frac{\partial^2 l(\cdot)}{\left(\partial\sigma_\mu^2\right)^2}
\end{bmatrix} \qquad (27)
$$

where $l\left(\cdot\right)$ is the log-likelihood function given by

$$
l\left(\cdot\right)=\text{constant}-\frac{1}{2}\log\left|\Sigma\right|-\frac{1}{2}\left(y-X\beta\right)'\Sigma^{-1}\left(y-X\beta\right) \qquad (28)
$$

where for convenience we have set,

$$
\Sigma=\sigma_\nu^2A+\sigma_\mu^2B \text{ with } A=I_N\otimes\Gamma, B=I_N\otimes\iota_T\iota_T'
$$

The first partial derivatives are obtained as,

$$
\begin{cases}
\dfrac{\partial l(\cdot)}{\partial \beta} = X'\Sigma^{-1}(y - X\beta) = 0 \\
\dfrac{\partial l(\cdot)}{\partial \rho} = -\dfrac{\sigma_{\nu}^{2}}{2} tr\left(\Sigma^{-1}A_{\rho}\right) + \dfrac{\sigma_{\nu}^{2}}{2} u'\left(\Sigma^{-2}A_{\rho}\right)u = 0 \\
\dfrac{\partial l(\cdot)}{\partial \sigma_{\nu}^{2}} = -\dfrac{1}{2} tr\left(\Sigma^{-1}A\right) + \dfrac{1}{2} u'\left(\Sigma^{-2}A\right)u = 0 \\
\dfrac{\partial l(\cdot)}{\partial \sigma_{\mu}^{2}} = -\dfrac{1}{2} tr\left(\Sigma^{-1}B\right) + \dfrac{1}{2} u'\left(\Sigma^{-2}B\right)u = 0
\end{cases} \tag{29}
$$

with

$$
A_{\rho} = \frac{\partial A}{\partial \rho} = I_N \otimes \Gamma_{\rho} \text{ and } \Gamma_{\rho} = \frac{\partial \Gamma}{\partial \rho} \tag{30}
$$

The second partial derivatives are therefore,

$$
\begin{cases}
\dfrac{\partial^2 l(\cdot)}{\partial \beta \partial \beta'} = X'\Sigma^{-1}X \\
\dfrac{\partial^2 l(\cdot)}{(\partial \rho)^2} = -\dfrac{\sigma_{\nu}^{2}}{2} tr\left(-\sigma_{\nu}^{2}\Sigma^{-2}A_{\rho}^{2} + \Sigma^{-1}\dfrac{\partial A_{\rho}}{\partial \rho}\right) + \dfrac{\sigma_{\nu}^{2}}{2} u'\left(-2\sigma_{\nu}^{2}\Sigma^{-3}A_{\rho}^{2} + \Sigma^{-2}\dfrac{\partial A_{\rho}}{\partial \rho}\right)u \\
\dfrac{\partial^2 l(\cdot)}{\left(\partial \sigma_{\nu}^{2}\right)^2} = \dfrac{1}{2} tr\left(\Sigma^{-2}A^{2}\right) - u'\left(\Sigma^{-3}A^{2}\right)u \\
\dfrac{\partial^2 l(\cdot)}{\left(\partial \sigma_{\mu}^{2}\right)^2} = \dfrac{1}{2} tr\left(\Sigma^{-2}B^{2}\right) - u'\left(\Sigma^{-3}B^{2}\right)u \\
\dfrac{\partial^2 l(\cdot)}{\partial \beta \partial \rho} = -\sigma_{\nu}^{2}X'\Sigma^{-2}A_{\rho}u \\
\dfrac{\partial^2 l(\cdot)}{\partial \beta \partial \sigma_{\nu}^{2}} = -X'\Sigma^{-2}Au \\
\dfrac{\partial^2 l(\cdot)}{\partial \beta \partial \sigma_{\mu}^{2}} = -X'\Sigma^{-2}Bu \\
\dfrac{\partial^2 l(\cdot)}{\partial \rho \partial \sigma_{\nu}^{2}} = -\dfrac{1}{2} tr\left(-\sigma_{\nu}^{2}\Sigma^{-2}A_{\rho}A + \Sigma^{-1}A_{\rho}\right) + \dfrac{1}{2} u'\left(-2\sigma_{\nu}^{2}\Sigma^{-3}A\rho A + \Sigma^{-2}A_{\rho}\right)u \\
\dfrac{\partial^2 l(\cdot)}{\partial \rho \partial \sigma_{\mu}^{2}} = \dfrac{\sigma_{\nu}^{2}}{2} tr\left(\Sigma^{-2}A_{\rho}B\right) - \sigma_{\nu}^{2}u'\left(\Sigma^{-3}A_{\rho}B\right)u \\
\dfrac{\partial^2 l(\cdot)}{\partial \sigma_{\nu}^{2} \partial \sigma_{\mu}^{2}} = \dfrac{1}{2} tr\left(\Sigma^{-2}BA\right) - u'\left(\Sigma^{-3}BA\right)u
\end{cases} \tag{31}
$$

Taking the expectations, one gets

$$\begin{cases} E\left(u'\left(\Sigma^{-3}A^2\right)u\right) = E\left[tr\left(u'\Sigma^{-3}A^2u\right)\right] = tr\left(A^2E\left(uu'\right)\Sigma^{-3}\right) = tr\left(A^2\Sigma^{-2}\right) \\ E\left(u'\left(\Sigma^{-3}B^2\right)u\right) = E\left[tr\left(u'\Sigma^{-3}B^2u\right)\right] = tr\left(B^2E\left(uu'\right)\Sigma^{-3}\right) = tr\left(B^2\Sigma^{-2}\right) \\ E\left(u'\left(-2\sigma_{\nu}^2\Sigma^{-3}A_{\rho}^2 + \Sigma^{-2}\frac{\partial A_{\rho}}{\partial\rho}\right)u\right) = 2\frac{\partial}{\partial\rho}\left[tr\left(\Sigma^{-1}A_{\rho}\right)\right] - tr\left(\Sigma^{-1}\frac{\partial A_{\rho}}{\partial\rho}\right) \\ E\left(u'\left(\Sigma^{-3}BA\right)u\right) = tr\left(\Sigma^{-2}BA\right) \\ E\left(X'\Sigma^{-2}A_{\rho}u\right) = E\left(X'\Sigma^{-2}Bu\right) = E\left(X'\Sigma^{-2}Au\right) = 0 \\ E\left(u'\left(-2\sigma_{\nu}^2\Sigma^{-3}A\rho A + \Sigma^{-2}A_{\rho}\right)u\right) = 2\frac{\partial}{\partial\rho}\left[tr\left(\Sigma^{-1}A\right)\right] - tr\left(\Sigma^{-1}A_{\rho}\right) \\ E\left(u'\left(\Sigma^{-3}A_{\rho}B\right)u\right) = tr\left(\Sigma^{-2}A_{\rho}B\right) \end{cases} \tag{32}$$

and therefore, one obtains

$$\begin{cases} E\left(\frac{\partial^2 l(\cdot)}{\partial\beta\partial\beta'}\right) = X'\Sigma^{-1}X \\ E\left(\frac{\partial^2 l(\cdot)}{(\partial\rho)^2}\right) = \frac{1}{2}\frac{\partial}{\partial\rho}\left[tr\left(\Sigma^{-1}A_{\rho}\right)\right] - \frac{1}{2}tr\left(\Sigma^{-1}\frac{\partial A_{\rho}}{\partial\rho}\right) \\ E\left(\frac{\partial^2 l(\cdot)}{\left(\partial\sigma_{\nu}^2\right)^2}\right) = \frac{1}{2}tr\left(\Sigma^{-2}A^2\right) = \frac{1}{2}\frac{\partial}{\partial\sigma_{\nu}^2}\left[tr\left(\Sigma^{-1}A\right)\right] \\ E\left(\frac{\partial^2 l(\cdot)}{\left(\partial\sigma_{\mu}^2\right)^2}\right) = \frac{1}{2}tr\left(\Sigma^{-2}B^2\right) = \frac{1}{2}\frac{\partial}{\partial\sigma_{\mu}^2}\left[tr\left(\Sigma^{-1}B\right)\right] \\ E\left(\frac{\partial^2 l(\cdot)}{\partial\beta\partial\rho}\right) = 0 \\ E\left(\frac{\partial^2 l(\cdot)}{\partial\beta\partial\sigma_{\nu}^2}\right) = 0 \\ E\left(\frac{\partial^2 l(\cdot)}{\partial\beta\partial\sigma_{\mu}^2}\right) = 0 \\ E\left(\frac{\partial^2 l(\cdot)}{\partial\rho\partial\sigma_{\nu}^2}\right) = \frac{1}{2}\frac{\partial}{\partial\rho}\left[tr\left(\Sigma^{-1}A\right)\right] - \frac{1}{2}tr\left(\Sigma^{-1}A_{\rho}\right) \\ E\left(\frac{\partial^2 l(\cdot)}{\partial\rho\partial\sigma_{\mu}^2}\right) = -\frac{1}{2}tr\left(\Sigma^{-2}A_{\rho}B\right) = \frac{1}{2}\frac{\partial}{\partial\rho}\left[tr\left(\Sigma^{-1}B\right)\right] \\ \frac{\partial^2 l(\cdot)}{\partial\sigma_{\nu}^2\partial\sigma_{\mu}^2} = -\frac{1}{2}tr\left(\Sigma^{-2}BA\right) = \frac{1}{2}\frac{\partial}{\partial\sigma_{\mu}^2}\left[tr\left(\Sigma^{-1}A\right)\right] \end{cases} \tag{33}$$

It then appears that the above expressions can easily be computed as soon as the following variables are known, say

$$tr\left(\Sigma^{-1}A\right),\ tr\left(\Sigma^{-1}B\right),\ tr\left(\Sigma^{-1}A_{\rho}\right),\ tr\left(\Sigma^{-1}\frac{\partial A_{\rho}}{\partial\rho}\right)$$

The first expression $tr\left(\Sigma^{-1}A\right)$ has a simple form. We have,

$$\begin{aligned}
tr\left(\Sigma^{-1}A\right) &= \frac{1}{\sigma_{\varepsilon}^{2}}tr\left[\left(I_N\otimes C'C\Gamma\right)-\frac{\sigma_{\mu}^{2}}{\sigma_{\varepsilon}^{2}+d\sigma_{\mu}^{2}}\left(I_N\otimes C'\iota_T^{\alpha}\iota_T^{\alpha'}C\Gamma\right)\right]\\
&= tr\left[\frac{1}{\sigma_{\nu}^{2}}\left(I_N\otimes I_T\right)-\frac{\sigma_{\mu}^{2}}{\sigma_{\varepsilon}^{2}+d\sigma_{\mu}^{2}}\left(I_N\otimes C\iota_T\iota_T'C'\right)\right]\\
&= \frac{N}{\sigma_{\nu}^{2}}\left(T-d\eta\right) \qquad (34)
\end{aligned}$$

The second expression $tr\left(\Sigma^{-1}B\right)$ is also straightforward. We get,

$$\begin{aligned}
tr\left(\Sigma^{-1}B\right) &= \frac{1}{\sigma_{\varepsilon}^{2}}tr\left[\left(I_N\otimes C'C\iota_T\iota_T'\right)-\frac{\sigma_{\mu}^{2}}{\sigma_{\varepsilon}^{2}+d\sigma_{\mu}^{2}}\left(I_N\otimes C'C\iota_T\iota_T'C'C\iota_T\iota_T'\right)\right]\\
&= \frac{Nd}{\sigma_{\varepsilon}^{2}}\left(1-\frac{d\sigma_{\mu}^{2}}{\sigma_{\varepsilon}^{2}+d\sigma_{\mu}^{2}}\right)=\frac{Nd}{\sigma_{\varepsilon}^{2}+d\sigma_{\mu}^{2}}\\
&= \frac{Nd}{\sigma_{\mu}^{2}}\eta \qquad (35)
\end{aligned}$$

The third expression is much more involved. We obtain [See Appendix B for details],

$$tr\left(\Sigma^{-1}A_{\rho}\right)=\frac{-2N\left(T-1\right)}{\left(1-\rho^{2}\right)\sigma_{\nu}^{2}}\left(\rho+\left(1-\rho\right)^{2}\eta\right) \qquad (36)$$

Finally, we derive the last expression $tr\left(\Sigma^{-1}\frac{\partial A_{\rho}}{\partial\rho}\right)$ [See Appendix C for details] .

$$tr\left(\Sigma^{-1}\frac{\partial A_{\rho}}{\partial\rho}\right)=-\frac{4N\eta}{\left(1-\rho^{2}\right)\sigma_{\nu}^{2}}\left[\left(1-\rho\right)\left(T-1\right)-\left(1-\rho^{T-1}\right)\right] \qquad (37)$$

We can now get the information matrix $J\left(\gamma\right)$ as

$$J\left(\gamma\right)=\begin{pmatrix}V_{\beta}^{-1} & 0\\ 0 & V_{\theta}^{-1}\end{pmatrix} \qquad (38)$$

where

$$V_{\beta} = E\left[\left(\hat{\beta} - \beta\right)\left(\hat{\beta} - \beta\right)'\right] = \left(X'\Sigma^{-1}X\right)^{-1}$$

and

$$V_{\theta} = \begin{pmatrix} M_1 & M_2 & M_3 \\ M_2 & M_4 & M_5 \\ M_3 & M_5 & M_6 \end{pmatrix}^{-1} = E\left[\left(\hat{\theta} - \theta\right)\left(\hat{\theta} - \theta\right)'\right], \theta = \left(\rho, \sigma_{\nu}^2, \sigma_{\mu}^2\right)$$

with

$$\begin{aligned} M_1 = & \frac{N(T-1)(1+\rho^2)}{(1-\rho^2)^2 \sigma_{\nu}^2} \\ & - \frac{2N}{(1+\rho)\sigma_{\nu}^2}\left(\frac{T-1}{1+\rho} + T - 1 - \frac{1-\rho^{T-1}}{1-\rho}\right)\eta \\ & + \frac{2N(T-1)(1-\rho)}{(1+\rho)\sigma_{\mu}^2\sigma_{\nu}^2}\left(\rho\sigma_{\nu}^2 + \left(\frac{d}{1-\rho} - 1\right)\sigma_{\mu}^2\right)\eta^2 \end{aligned} \tag{39a}$$

$$\begin{aligned} M_2 = & -\frac{N(T-1)\rho}{(1-\rho^2)\sigma_{\nu}^2} \\ & - \frac{N}{\sigma_{\nu}^2}\left[\frac{d}{1-\rho} - 1 + \frac{(T-1)(1-\rho)}{1+\rho}\right]\eta \\ & + \frac{Nd}{\sigma_{\mu}^2\sigma_{\nu}^2}\left(\rho\sigma_{\nu}^2 + \left(\frac{d}{1-\rho} - 1\right)\sigma_{\mu}^2\right)\eta^2 \end{aligned} \tag{39b}$$

$$M_3 = \frac{N}{\sigma_{\mu}^2}\left(\frac{d}{1-\rho} - 1\right)\eta - \frac{Nd}{\sigma_{\mu}^4}\left(\rho\sigma_{\nu}^2 + \left(\frac{d}{1-\rho} - 1\right)\sigma_{\mu}^2\right)\eta^2 \tag{39c}$$

$$M_4 = \frac{NT}{2\sigma_{\nu}^4} - \frac{Nd}{2\sigma_{\nu}^4}\eta - \frac{Nd(1-\rho^2)}{2\sigma_{\nu}^2\sigma_{\mu}^2}\eta^2 \tag{39d}$$

$$M_5 = \frac{Nd(1-\rho^2)}{2\sigma_{\mu}^4}\eta^2 \tag{39e}$$

and

$$M_6 = \frac{Nd^2}{2\sigma_{\mu}^4}\eta^2 \tag{39f}$$

It is worth mentioning that all the second order moments in $\theta$ are quadratic functions of $\eta = \dfrac{\sigma_{\mu}^2}{\sigma_{\varepsilon}^2 + d\sigma_{\mu}^2}$ [See Appendix D for details].

Also in the expression given by equation (17),

$$w^{'}\Sigma^{-1}u = F^{'}u_i$$

$F^{'} = \begin{pmatrix} f\, g \ldots g\, h \end{pmatrix}$ where $f$, $g$, and $h$ are non linear functions of $\theta$. Using the first order approximation, we have

$$\hat{F} - F \approx \Lambda\left(\hat{\theta} - \theta\right) \tag{40}$$

with

$$\Lambda = \begin{pmatrix} \dfrac{\partial f}{\partial \rho} & \dfrac{\partial f}{\partial \sigma_\nu^2} & \dfrac{\partial f}{\partial \sigma_\mu^2} \\ \dfrac{\partial g}{\partial \rho} & \dfrac{\partial g}{\partial \sigma_\nu^2} & \dfrac{\partial g}{\partial \sigma_\mu^2} \\ \vdots & \vdots & \vdots \\ \dfrac{\partial g}{\partial \rho} & \dfrac{\partial g}{\partial \sigma_\nu^2} & \dfrac{\partial g}{\partial \sigma_\mu^2} \\ \dfrac{\partial h}{\partial \rho} & \dfrac{\partial h}{\partial \sigma_\nu^2} & \dfrac{\partial h}{\partial \sigma_\mu^2} \end{pmatrix} \tag{41}$$

and

$$\begin{cases} f = \dfrac{\sigma_\mu^2\left(1-\rho^h\right)}{(1+\rho)\,\sigma_\nu^2 + \sigma_\mu^2\left[1+\rho+(T-1)(1-\rho)\right]} \\ g = \dfrac{\sigma_\mu^2\left(1-\rho^h\right)(1-\rho)}{(1+\rho)\,\sigma_\nu^2 + \sigma_\mu^2\left[1+\rho+(T-1)(1-\rho)\right]} \\ h = \rho^h + \dfrac{\sigma_\mu^2\left(1-\rho^h\right)}{(1+\rho)\,\sigma_\nu^2 + \sigma_\mu^2\left[1+\rho+(T-1)(1-\rho)\right]} \end{cases} \tag{42}$$

We can now derive the targeted AMSE, knowing that

$$\begin{cases} E\left[\left(x_{i,T+h} - F^{'}x_i\right)^{'}\left(\hat{\beta}-\beta\right)\left(\hat{\beta}-\beta\right)^{'}\left(x_{i,T+h} - F^{'}x_i\right)\right] = \\ \left(x_{i,T+h} - F^{'}x_i\right)^{'} V_{\beta}\left(x_{i,T+h} - F^{'}x_i\right) \\ E\left[\left(\hat{F}-F\right)^{'}u_i\right]^2 = E\left[\left(\hat{F}-F\right)^{'}(y_i - x_i\beta)\right]^2 = \beta^{'}x_i\Lambda V_{\theta}\Lambda^{'}x_i^{'}\beta \end{cases} \tag{43}$$

where we assume that the $y_{it}$ observations are independent of the estimators of standard deviations. Also, since $\hat{\beta}$ and $\theta$ are independent, the expectation of all cross terms involving $\hat{\beta}-\beta$ and $\hat{F}-F$ is equal to zero. Therefore, we have

$$AMSE\left(\widehat{\widehat{y}}_{i,T,h}\right) = \sigma_{\nu}^{2}\left(1-\rho^{2h}+\frac{d\left(1+\rho\right)}{1-\rho}f^{2}\right)+\left(1-\rho^{h}-\left(T-\left(T-2\right)\rho\right)f\right)^{2}\sigma_{\mu}^{2}$$
$$+\left(x_{i,T+h}-F^{'}x_{i}\right)^{'}V_{\beta}\left(x_{i,T+h}-F^{'}x_{i}\right)+\beta^{'}x_{i}\Lambda V_{\theta}\Lambda^{'}x_{i}^{'}\beta \quad (44)$$

### 3.3 The AMSE for the Truncated Predictor

The truncated predictor as defined above is,

$$y_{i,T,h}^{*} = x_{i,T+h}^{\prime}\hat{\beta}$$

This predictor uses efficient parameters' estimators but ignores the last term in the optimal predictor. The associated prediction error is

$$y_{i,T,h} - y_{i,T,h}^{*} = x_{i,T+h}^{\prime}\left(\beta-\hat{\beta}\right)+u_{i,T+h} \quad (45)$$

$$E\left(y_{i,T,h}-y_{i,T,h}^{*}\right)^{2} = x_{i,T+h}^{\prime}V_{\beta}x_{i,T+h}+E\left(u_{i,T+h}^{2}\right)-2x_{i,T+h}^{\prime}E\left[\left(\hat{\beta}-\beta\right)u_{i,T+h}\right].$$

And since

$$E\left[\left(\hat{\beta}-\beta\right)u_{i,T+h}\right] = \left(X^{'}\Sigma^{-1}X\right)^{-1}X^{'}\Sigma^{-1}E\left(uu_{i,T+h}\right)$$
$$= \left(X^{'}\Sigma^{-1}X\right)^{-1}X^{'}F \quad (46)$$

Hence,

$$AMSE\left(y_{i,T,h}^{*}\right) = x_{i,T+h}^{\prime}V_{\beta}x_{i,T+h}+\sigma_{\nu}^{2}+\sigma_{\mu}^{2}-2x_{i,T+h}^{\prime}\left(X^{'}\Sigma^{-1}X\right)^{-1}X^{'}F \quad (47)$$

### 3.4 AMSE for the Misspecified Predictor

The misspecified predictor is given by

$$y_{i,T,h}^{**} = x_{i,T+h}^{\prime}\hat{\beta}_{OLS}$$

The above predictor ignores autorrelation and prediction errors in the estimation of the one way error component model. The predictor error is given by

$$y_{i,T,h} - y_{i,T,h}^{**} = x_{i,T+h}^{\prime}\left(\beta-\hat{\beta}_{OLS}\right)+u_{i,T+h} \quad (48)$$

$$E\left(y_{i,T+h}-y_{i,T,h}^{**}\right)^{2} = x_{i,T+h}^{\prime}Var\left(\hat{\beta}_{OLS}\right)x_{i,T+h}+E\left(u_{i,T+h}^{2}\right)$$
$$-2x_{i,T+h}^{\prime}E\left[\left(\beta-\hat{\beta}_{OLS}\right)u_{i,T+h}\right]$$

with

$$Var\left(\hat{\beta}_{OLS}\right)=\left(X'X\right)^{-1}X'\Sigma X\left(X'X\right)^{-1} \tag{49}$$

and

$$E\left[\left(\beta-\hat{\beta}_{OLS}\right)u_{i,T+h}\right]=-E\left[\left(X'X\right)^{-1}X'uu_{i,T+h}\right]=-\left(X'X\right)^{-1}X'w$$

Hence,

$$\begin{aligned}AMSE\left(y^{**}_{i,T,h}\right)=&\,x'_{i,T+h}\left(X'X\right)^{-1}X'\Sigma X\left(X'X\right)^{-1}x_{i,T+h}+\sigma^2_\nu+\sigma^2_\mu\\&-2x'_{i,T+h}\left(X'X\right)^{-1}X'w\end{aligned} \tag{50}$$

## 3.5 AMSE for the Fixed Effect Predictor

The fixed effect model predictor at horizon $h\geq 1$ is given by, $\tilde{y}_{i,T,h}=x'_{i,T+h}\tilde{\beta}+\tilde{\mu}_i$. In fact, we assume that the $\mu_i$ are fixed parameters to be estimated. The predictor error is given by,

$$\begin{aligned}y_{i,T,h}-\tilde{y}_{i,T,h}=&\,x'_{i,T+h}\left(\beta-\tilde{\beta}\right)+u_{i,T+h}-\tilde{u}_i=x'_{i,T+h}\left(\beta-\tilde{\beta}\right)+u_{i,T+h}-\left(\bar{y}_{i.}-\tilde{\beta}'\bar{x}_{i.}\right)\\=&\left(x_{i,T+h}-\bar{x}_{i.}\right)'\left(\beta-\tilde{\beta}\right)+u_{i,T+h}-\bar{u}_i\end{aligned} \tag{51}$$

Moreover,

$$Var\left(\tilde{\beta}\right)=\sigma^2_\nu\left(X'QX\right)^{-1}X'Q\left(I_N\otimes\Gamma\right)QX\left(X'QX\right)^{-1}$$

whereas

$$\begin{cases}E\left[\left(\beta-\tilde{\beta}\right)\bar{u}_i\right]=-\left(X'QX\right)^{-1}X'QE\left(u\bar{u}_i\right)=-\dfrac{1}{T}\left(X'QX\right)^{-1}X'Q\Sigma\left(d_i\otimes\iota_T\right)\\E\left[\left(\beta-\tilde{\beta}\right)u_{i,T+h}\right]=-\left(X'QX\right)^{-1}X'QE\left(uu_{i,T+h}\right)=-\left(X'QX\right)^{-1}X'Qw\\E\left(\bar{u}_iu_{i,T+h}\right)=\dfrac{1}{T}\left(d'_i\otimes\iota'_T\right)E\left(uu_{i,T+h}\right)=\dfrac{1}{T}\left(d'_i\otimes\iota'_T\right)w\\E\left(\bar{u}'_i\bar{u}_i\right)=\dfrac{1}{T^2}\left(d'_i\otimes\iota'_T\right)\Sigma\left(d_i\otimes\iota_T\right)=\dfrac{\sigma^2_\nu}{T^2}\iota'_T\Gamma\iota_T+\sigma^2_\mu\end{cases} \tag{52}$$

Consequently,

$$\begin{aligned}AMSE\left(\tilde{y}_{i,T,h}\right)=&\,\sigma^2_\nu\left(x_{i,T+h}-\bar{x}_{i.}\right)'\left(X'QX\right)^{-1}X'Q\left(I_N\otimes\Gamma\right)QX\left(X'QX\right)^{-1}\left(x_{i,T+h}-\bar{x}_{i.}\right)\\&-2\left(x_{i,T+h}-\bar{x}_{i.}\right)'\left(X'QX\right)^{-1}X'Qw\\&+\frac{2}{T}\left(x_{i,T+h}-\bar{x}_{i.}\right)'\left(X'QX\right)^{-1}\Sigma X'Q\left(d_i\otimes\iota_T\right)\\&-\frac{2}{T}\left(d'_i\otimes\iota'_T\right)w+\sigma^2_\nu+\frac{\sigma^2_\nu}{T^2}\iota'_T\Gamma\iota_T+2\sigma^2_\mu\end{aligned} \tag{53}$$

# 4 Monte Carlo results

To determine the accuracy of the asymptotic approximation of the MSE of the predictors, some Monte Carlo experiments were conducted. The data are generated from the simple regression

$$y_{it} = \alpha + \beta x_{it} + u_{it}, \qquad i = 1, \ldots, N; \quad t = 1, \ldots, T + h$$

with one way error components, $u_{it} = \mu_i + \nu_{it}$ and $\nu_{it} = \rho\nu_{i,t-1} + \varepsilon_{it}$. Throughout the experiment the parameters were set at $\alpha = 5$, $\beta = 0.5$, with the total variance $\sigma^2 = \sigma_\mu^2 + \sigma_\nu^2$ fixed at 30. The variable $x_{it}$ was generated as in Nerlove [1971] with $x_{it} = 0.1t + 0.5x_{i,t-1} + \omega_{it}$ where contrary to Nerlove_[1971] $\omega_{it}$ is a random variable normally distributed, i.e., $\omega_{it} \sim N(0, 1)$ and $x_{i0} = 5 + 10\omega_{i0}$. The first 20 period observations were discarded to minimize the effect of initial values. Predictions were made for only $(h = 1)$ one period ahead. It is worth mentioning that the predictor for this panel data model changes with $h$ only through $x_{i,T+h}$. In fact, it is the presence of the same individual in the panel that creates the correlation over time, and this is a constant correlation that does not die out no matter how far ahead we are predicting. In order to depict the typical labor or consumer panel where $N$ is large and $T$ is small, the sample sizes in the different experiments were chosen as $N = 50$, 100, 500 and $T = 10$, 20, 50, with 1,000 replications performed for each experiment. For each replication, $(N(T + h) + N)$ $NID(0, 1)$ random numbers are generated. The first $N$ $(T + h)$ random numbers were used to generate the $\nu_{it}$s from $NID\left(0, \sigma_\nu^2\right)$ and the remaining $N$ random numbers are used to generate the $\mu_i$s from $NID\left(0, \sigma_\mu^2\right)$. With this design and $\sigma^2 = \sigma_\nu^2 + \sigma_\mu^2$ the implied values of $\phi = \frac{\sigma_\mu^2}{\sigma^2}$ are 0.01, 0.3, 0.6, and 0.9. For each of the predictor considered in this paper, the AMSE for a one-step ahead prediction was computed from the formulas derived previously, and the sampling MSE was computed as,

$$\text{MSE} = \frac{1}{NR}\sum_{r=1}^{R}\sum_{i=1}^{N}\left(y_{i,T+h} - \widehat{\widehat{y}}_{i,T,h}\right)^2$$

Following Spitzer and Baillie [1980], the quantity

$$\text{AMSE BIAS VARIANCE} = \frac{1}{NR}\sum_{r=1}^{R}\sum_{i=1}^{N}\left[\left(y_{i,T+h} - \widehat{\widehat{y}}_{i,T,h}\right)^2 - \text{AMSE}\left(\widehat{\widehat{y}}_{i,T,h}\right)\right]^2$$

were the summation extends over all $R = 1,000$ replications and $N$ individuals for each $(T, \rho, \phi, s)$. On defining $q$ =MSE−AMSE, it is possible to test $H_0 : q = 0$ versus $H_1 : q \neq 0$ by using the statistic

$$Z = \frac{\sqrt{R}q}{(\text{AMSE BIAS VARIANCE})^{\frac{1}{2}}}$$

Since $N$ is fixed for each particular experiment, then for large $R$, it can be seen that $Z$ will be approximately distributed as $N(0,1)$ since AMSE BIAS VARIANCE is an estimate of the population variance of $q$ and both MSE and AMSE are $\chi^2(1)$ variables.

Table 1 reports the results of MSE and AMSE for the ordinary, truncated, misspecified and fixed effects predictors averaged over all individuals replications. The results can be summarized as follows:

(i) The sampling results are very close to the analytical ones as evident by the closeness of MSE to AMSE for all experiments and predictors considered. The MSE of the ordinary and fixed effect predictor tend to understate their AMSE, particularly for $\rho = 0.01$, but these differences are not statistically significant. In fact, the difference between MSE and AMSE is insignificant for all experiments and predictors.

(ii) When the autocorrelation coefficient is close to zero, i.e. $\rho = 0.01$, both the analytical and sampling results in Table 1 show that there are substantial gains in mean squared error prediction by using the ordinary predictor instead of the misspecified or the truncated predictors, for all values of $\phi$. The reduction in MSE is about tenfold for $\rho = 0.9$ and a little more than twofold for $\rho = 0.6$ for various values of $N$ and $T$. This result is in accordance with the one obtained in Baillie and Baltagi [1999]. The ordinary predictor for the error component model comprehensively outperforms the truncated and misspecified predictors and is recommended when predicting with panel data with no autocorrelation.

(iii) When the autocorrelation coefficient starts increasing, the ordinary predictor for the error component model is not necessarily the preferred one. Particularly, for high values of $\phi$ (0.6, 0.9) and high values of $\rho$ (0.3, 0.6, 0.9) the fixed effects predictor performs better than the ordinary predictor, being a close second to the ordinary predictor for all experiments. The MSE and AMSE of the fixed effects and the ordinary predictors decrease by tenfold as $\rho$ increases from 0 to 0.9. This is true for all values of $N$ and $T$ considered. This result has been obtained by Yamamoto (1979) in the case of regression analysis.

(iv) For $\rho = 0$ and $\phi$ close to 0 (0.01) both truncated and misspecified predictors results are acceptable when compared to the ordinary and fixed effects predictors. However, as $\phi$ increases, the truncated and misspecified predictors drastically worsen.

____[INSERT TABLE 1 AROUND HERE]____

## 5 Final Remarks

This paper has derived the asymptotic MSE of the ordinary and fixed effects predictors in the context of the one-way error component model with AR(1) errors in the disturbance term. Simulation evidence has confirmed the adequacy of the asymptotic approximation in realistic sample sizes and indicates the importance of allowing for parameter uncertainty when forming prediction confidence intervals or when undertaking ex ante prediction stability testing.

In accordance with previously derived results, it appears that the optimal predictor outperforms the other predictors when $\rho$ and $\phi$ display low values. For high values of $\rho$ and $\phi$, the fixed effects predictor in the context of the one way setting with autocorrelation, performs well being a close second to the ordinary predictor for all experiments.

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# Appendix A: Generalizing Baltagi and Li [1992] and Baltagi [2008] Results

Baltagi and Li [1992] and Baltagi [2008] have derived the one period ahead $i$th individual prediction in a one-way AR(1) serial correlation context. Using our notations, we have

$$\hat{y}_{i,T+1} = x'_{i,T+1}\hat{\beta}_{GLS} + w'\Sigma^{-1}u$$

with

$$w'\Sigma^{-1}u = \rho u_{i,T} + \frac{(1-\rho)^2 \sigma_\mu^2}{\sigma_\alpha^2}\left(\alpha u_{i,1}^* + \sum_{t=2}^{T} u_{it}^*\right) \tag{A-1}$$

where $u_{it}^{*}s$ are the Prais-Winsten transformed errors, $\alpha = \sqrt{\dfrac{1+\rho}{1-\rho}}$ and $\sigma_{\alpha}^{2} = \left(\alpha^{2}+T-1\right)\sigma_{\mu}^{2}\left(1-\rho\right)^{2}+\sigma_{\varepsilon}^{2}$. One gets

$$u_{i}^{*} = Cu_{i} = \begin{pmatrix} \sqrt{1-\rho^{2}}u_{i1} \\ u_{i2}-\rho u_{i1} \\ \vdots \\ u_{i,T-1}-\rho u_{i,T-2} \\ u_{iT}-\rho u_{i,T-1} \end{pmatrix} \tag{A-2}$$

In terms of a one-step procedure, one has

$$\begin{aligned} w^{'}\Sigma^{-1}u &= \rho u_{i,T} + \left(\frac{\left(1-\rho\right)^{2}\sigma_{\mu}^{2}}{\sigma_{\alpha}^{2}}\right)\left[\sqrt{\frac{1+\rho}{1-\rho}}\sqrt{1-\rho^{2}}u_{i1}+\sum_{t=2}^{T}\left(u_{it}-\rho u_{i,t-1}\right)\right] \\ &= \rho u_{i,T} + m\left(1+\rho\right)u_{i1}+m\left(1-\rho\right)\sum_{t=2}^{T-1}u_{it}+mu_{iT}-\rho m u_{i1} \end{aligned} \tag{A-3}$$

with

$$m = \frac{\left(1-\rho\right)^{2}\sigma_{\mu}^{2}}{\left[1-\rho^{2}+\left(1-\rho\right)^{2}\left(T-1\right)\right]\sigma_{\mu}^{2}+\sigma_{\varepsilon}^{2}} \tag{A-4}$$

Hence,

$$w^{'}\Sigma^{-1}u = mu_{i1}+m\left(1-\rho\right)\sum_{t=2}^{T-1}u_{it}+\left(\rho+m\right)u_{i,T} \tag{A-5}$$

Our goal in this appendix is to show that the Baltagi and Li [1992] and Baltagi [2008] formula is actually a specific case of equation (18) which is a $h$ period ahead prediction formula. When $h=1$, equation (17) becomes

$$w^{'}\Sigma^{-1}u = F^{'}u_{i} = \left(f\, g \ldots g\, h\right)u_{i} = fu_{i1}+g\sum_{t=2}^{T-1}u_{it}+hu_{i,T}$$

where $f$, $g$, and $h$ are given by equation (16). We only need to show that, when $h=1$, we have $f=m$, $g=m\left(1-\rho\right)$, and $h=\rho+m$.

By definition, for $h=1$,

$$f = \frac{\sigma_{\mu}^{2}}{\sigma_{\varepsilon}^{2}+d\sigma_{\mu}^{2}}\left(1-\rho\right)\left(1-\rho\right) = \frac{\sigma_{\mu}^{2}}{\sigma_{\varepsilon}^{2}+d\sigma_{\mu}^{2}}\left(1-\rho\right)^{2}$$

Knowing that $d=\iota_{T}^{\alpha'}\iota_{T}^{\alpha}=1-\rho^{2}+\left(T-1\right)\left(1-\rho\right)^{2}$, one gets

$$f = \frac{\sigma_{\mu}^{2}}{\sigma_{\varepsilon}^{2}+\left[1-\rho^{2}+\left(1-\rho\right)^{2}\left(T-1\right)\right]\sigma_{\mu}^{2}}\left(1-\rho\right)^{2} = m$$

One also has, for $h = 1$,

$$g = \frac{\sigma_\mu^2}{\sigma_\varepsilon^2 + d\sigma_\mu^2} (1-\rho)(1-\rho)^2 = (1-\rho) f = (1-\rho) m$$

Lastly, for $h = 1$,

$$h = \rho + \frac{\sigma_\mu^2}{\sigma_\varepsilon^2 + d\sigma_\mu^2} (1-\rho)(1-\rho) = \rho + f = \rho + m$$

As a conclusion, equation (18) generalizes the findings of Baltagi and Li [1992] and Baltagi [2008] as a prediction formula in the AR(1) one-way error commponents model for $h$ periods ahead ($h \geq 1$).

## Appendix B: Deriving $\mathrm{Tr}\left(\Sigma^{-1} A_\rho\right)$

We are interested in computing $tr\left(\Sigma^{-1} A_\rho\right)$. From the definitions of $\Sigma^{-1}$ and $A_\rho$, it comes that

$$tr\left(\Sigma^{-1} A_\rho\right) = \frac{N}{\sigma_\varepsilon^2} tr\left(C' C \Gamma_\rho\right) - \frac{N\sigma_\mu^2}{\sigma_\varepsilon^2\left(\sigma_\varepsilon^2 + d\sigma_\mu^2\right)} tr\left[\left(C' C \iota_T \iota_T'\right)\left(C' C \Gamma_\rho\right)\right] \tag{B-1}$$

where, on the one hand,

$$C' C \Gamma_\rho = (a_{ij})_{\substack{i=1,\dots,T \\ j=1,\dots,T}}$$

with

$$a_{ij} = \begin{cases} i = 1 : a_{11} = -\rho, \; a_{1j} = \rho^{j-2}, \text{ for } j \geq 2 \\ 1 < i < T : a_{ij} = (1-\rho^2)\,\rho^{|i-j|-1}, \text{ for } i \neq j, \; a_{ii} = -2\rho \\ i = T : \; a_{Tj} = \rho^{T-j-1}, \text{ for } j < T, \; a_{TT} = -\rho \end{cases} \tag{B-2}$$

and on the other hand,

$$C' C \iota_T \iota_T' = (b_{ij})_{\substack{i=1,\dots,T \\ j=1,\dots,T}}$$

with

$$b_{ij} = \begin{cases} i = 1, T : b_{ij} = 1 - \rho \\ 1 < i < T : b_{ij} = (1-\rho^2) \end{cases} \tag{B-3}$$

It then comes that

$$\left(C' C \iota_T \iota_T'\right)\left(C' C \Gamma_\rho\right) = (c_{ij})_{\substack{i=1,\dots,T \\ j=1,\dots,T}}$$

with

$$c_{ij} = \begin{cases} i = 1, T : c_{ij} = (1-\rho) \sum_{k=1}^{T} a_{kj}, \ j = 1, \ldots, T \\ i = 2, \ldots, T-1 : c_{ij} = (1-\rho)\, c_{1j} \end{cases} \tag{B-4}$$

Thus, we firstly get

$$tr\left(C^{'} C \Gamma_{\rho}\right) = -2\rho + (T-2)(-2\rho) = -2\rho\,(T-1) \tag{B-5}$$

Secondly, we have

$$tr\left[\left(C^{'} C \iota_{T} \iota_{T}^{'}\right)\left(C^{'} C \Gamma_{\rho}\right)\right] = \sum_{i=1}^{T} c_{ii} = 2c_{11} + (1-\rho) \sum_{j=2}^{T-1} c_{1j} \tag{B-6}$$

For $j = 2$,we can write

$$\begin{aligned} c_{1j} &= (1-\rho)\left[\left(1-\rho^2\right) \sum_{\substack{i=2 \\ i \neq j}}^{T-1} \rho^{|i-j|-1} + \rho^{T-1-J} + \rho^{j-2} - 2\rho\right] \\ &= (1-\rho)\left[\left(1-\rho^2\right)\left(\sum_{k=0}^{j-3} \rho^k + \sum_{k=0}^{T-j-2} \rho^k\right) + \rho^{T-1-J} + \rho^{j-2} - 2\rho\right] \\ &= (1-\rho)\left[\left(1-\rho^2\right)\left(\frac{1-\rho^{j-2}}{1-\rho} + \frac{1-\rho^{T-j-1}}{1-\rho}\right) + \rho^{T-1-J} + \rho^{j-2} - 2\rho\right] \\ &= (1-\rho)\left(2 - \rho^{j-1} - \rho^{T-j}\right) \end{aligned} \tag{B-7}$$

Hence,

$$\begin{aligned} \sum_{j=2}^{T-1} c_{1j} &= (1-\rho) \sum_{j=2}^{T-1} \left(2 - \rho^{j-1} - \rho^{T-j}\right) \\ &= 2(1-\rho)(T-2) - 2\left(1-\rho^{T-1}\right) + 2(1-\rho) \end{aligned}$$

$$(1-\rho) \sum_{j=2}^{T-1} c_{1j} = 2(1-\rho)^2 (T-1) - 2(1-\rho)\left(1-\rho^{T-1}\right) \tag{B-8}$$

For $j = 1, T$, we have

$$\begin{aligned} c_{11} &= (1-\rho) \sum_{i=1}^{T} a_{i1} = (1-\rho)\left(-\rho + \sum_{i=2}^{T-1} \left(1-\rho^2\right) \rho^{i-2} + \rho^{T-2}\right) \\ &= (1-\rho)\left[-\rho + \rho^{T-2} + (1+\rho)\left(1-\rho^{T-2}\right)\right] \end{aligned}$$

$$2c_{11} = 2\left(1-\rho\right)\left(1-\rho^{T-1}\right) \tag{B-9}$$

As a result, we obtain

$$tr\left[\left(C^{'}C\iota_{T}\iota_{T}^{'}\right)\left(C^{'}C\Gamma_{\rho}\right)\right] = 2\left(1-\rho\right)^{2}\left(T-1\right)$$

Equation (B-1) becomes

$$tr\left(\Sigma^{-1}A_{\rho}\right) = \frac{N}{\sigma_{\varepsilon}^{2}}\left[-2\rho\left(T-1\right)\right] - \frac{N\sigma_{\mu}^{2}}{\sigma_{\varepsilon}^{2}\left(\sigma_{\varepsilon}^{2}+d\sigma_{\mu}^{2}\right)}\left[2\left(1-\rho\right)^{2}\left(T-1\right)\right]$$

i.e.

$$tr\left(\Sigma^{-1}A_{\rho}\right) = \frac{-2N\left(T-1\right)}{\left(1-\rho^{2}\right)\sigma_{\nu}^{2}}\left(\rho+\left(1-\rho\right)^{2}\eta\right)$$

which is known as equation (36), where $\eta = \dfrac{\sigma_{\mu}^{2}}{\sigma_{\varepsilon}^{2}+d\sigma_{\mu}^{2}}$.

# Appendix C: Deriving Tr$\left(\Sigma^{-1}\dfrac{\partial A_{\rho}}{\partial\rho}\right)$

We are interested in computing $tr\left(\Sigma^{-1}\dfrac{\partial A_{\rho}}{\partial\rho}\right)$ where $\dfrac{\partial A_{\rho}}{\partial\rho} = I_{N}\otimes\dfrac{\partial\Gamma_{\rho}}{\partial\rho}$. The definitions of $\Sigma^{-1}$ and $A_{\rho}$ yield

$$\frac{\partial\Gamma_{\rho}}{\partial\rho} = \left(\left|i-j\right|\left(\left|i-j\right|-1\right)\rho^{\left|i-j\right|-2}\right)_{\substack{i=1,\dots,T\\ j=1,\dots,T}} \tag{C-1}$$

so that

$$tr\left(\Sigma^{-1}\frac{\partial A_{\rho}}{\partial\rho}\right) = \frac{N}{\sigma_{\varepsilon}^{2}}tr\left(C^{'}C\Gamma_{\rho}\right) - \frac{N\sigma_{\mu}^{2}}{\sigma_{\varepsilon}^{2}\left(\sigma_{\varepsilon}^{2}+d\sigma_{\mu}^{2}\right)}tr\left[\left(C^{'}C\iota_{T}\iota_{T}^{'}\right)\left(C^{'}C\frac{\partial\Gamma_{\rho}}{\partial\rho}\right)\right] \tag{C-2}$$

We firstly notice that

$$C^{'}C\frac{\partial\Gamma_{\rho}}{\partial\rho} = \left(h_{ij}\right)_{\substack{i=1,\dots,T\\ j=1,\dots,T}}$$

with

$$h_{ij} = \begin{cases} i=1: \begin{cases} j=1,2:\ h_{1j}=0 \\ j>2:\ h_{1j}=2\left(j-2\right)\rho^{j-3} \end{cases} \\ 1<i<T: \begin{cases} i=j,\ h_{ij}=0 \\ \left|i-j\right|=1,\ h_{ij}=-2\rho \\ \left|i-j\right|>1,\ h_{ij}=2\left(\left|i-j\right|\left(1-\rho^{2}\right)-1\right)\rho^{\left|i-j\right|-2} \end{cases} \\ i=T: \begin{cases} j<T-1:\ h_{ij}=2\left(T-j-1\right)\rho^{T-j-2} \\ j=T-1,\ T-2:\ h_{ij}=0. \end{cases} \end{cases} \tag{C-3}$$

Thus,

$$tr\left(C^{'}C\frac{\partial\Gamma_{\rho}}{\partial\rho}\right)=0 \tag{C-4}$$

Secondly, if we set

$$\left(C^{'}C\iota_{T}\iota_{T}^{'}\right)\left(C^{'}C\frac{\partial\Gamma_{\rho}}{\partial\rho}\right)=\left(s_{ij}\right)_{\substack{i=1,\ldots,T\\ j=1,\ldots,T}} \tag{C-5}$$

we deduce that

$$tr\left[\left(C^{'}C\iota_{T}\iota_{T}^{'}\right)\left(C^{'}C\frac{\partial\Gamma_{\rho}}{\partial\rho}\right)\right]=\sum_{i=1}^{T}s_{ii}=(1-\rho)\,\varphi_{1}+(1-\rho)^{2}\sum_{j=2}^{T-1}\varphi_{j}+(1-\rho)\,\varphi_{T} \tag{C-6}$$

where,

$$\varphi_{j}=\sum_{i=1}^{T}h_{ij} \tag{C-7}$$

is the sum over the $j$th column of the matrix $C^{'}C\frac{\partial\Gamma_{\rho}}{\partial\rho}$. One gets,

$$\varphi_{1}=\varphi_{T}=2\,(1-\rho)\sum_{j=1}^{T-2}j\rho^{j-2} \tag{C-8}$$

and

$$\varphi_{j}=2\,(1-\rho)\left(\sum_{k=1}^{j-2}k\rho^{k-1}+\sum_{k=1}^{T-j-1}k\rho^{k-1}\right),\quad \text{for } j=2,\ldots,T-1. \tag{C-9}$$

It follows that,

$$\begin{aligned}\sum_{j=2}^{T-1}\varphi_{j}&=2\,(1-\rho)\left(\sum_{j=2}^{T-1}\sum_{k=1}^{j-2}k\rho^{k-1}+\sum_{j=2}^{T-1}\sum_{k=1}^{T-j-1}k\rho^{k-1}\right)\\&=4\,(1-\rho)\left(\sum_{k=1}^{1}k\rho^{k-1}+\sum_{k=1}^{2}k\rho^{k-1}+\cdots+\sum_{k=1}^{T-3}k\rho^{k-1}\right)\\&=4\,(1-\rho)\sum_{j=1}^{T-3}(T-2-j)\,j\rho^{j-1}\end{aligned} \tag{C-10}$$

Hence,

$$(1-\rho)^{2}\sum_{j=2}^{T-1}\varphi_{j}=4\,(1-\rho)^{2}\left(\sum_{j=1}^{T-3}(T-2-j)\,j\rho^{j-1}-\sum_{j=1}^{T-3}(T-2-j)\,j\rho^{j}\right)$$

Therefore,

$$tr\left[\left(C^{'}C\iota_T\iota_T^{'}\right)\left(C^{'}C\frac{\partial\Gamma_\rho}{\partial\rho}\right)\right]=4\left(1-\rho\right)^2\sum_{j=1}^{T-2}\left(T-j-1\right)\rho^{j-1} \qquad \text{(C-11)}$$

One can go further,

$$tr\left[\left(C^{'}C\iota_T\iota_T^{'}\right)\left(C^{'}C\frac{\partial\Gamma_\rho}{\partial\rho}\right)\right]=4\left(1-\rho\right)^2\left[\left(T-1\right)\frac{1-\rho^{T-2}}{1-\rho}-\sum_{j=1}^{T-2}j\rho^{j-1}\right]$$

with

$$\sum_{j=1}^{T-2}j\rho^{j-1}=\rho^{T-1}\frac{\left(T-1\right)\rho-\left(T-1\right)-\rho}{\rho\left(1-\rho\right)^2}+\frac{1}{\left(1-\rho\right)^2}$$

As a consequence,

$$\begin{aligned}tr\left[\left(C^{'}C\iota_T\iota_T^{'}\right)\left(C^{'}C\frac{\partial\Gamma_\rho}{\partial\rho}\right)\right]&=4\left(1-\rho\right)\left[\left(T-1\right)\left(1-\rho^{T-2}\right)+\left(T-1\right)\rho^{T-2}-\frac{1-\rho^{T-1}}{1-\rho}\right]\\&=4\left(1-\rho\right)\left(T-1\right)-4\left(1-\rho^{T-1}\right) \qquad \text{(C-12)}\end{aligned}$$

Equation (C-2) then becomes

$$tr\left(\Sigma^{-1}\frac{\partial A_\rho}{\partial\rho}\right)=-\frac{4N\sigma_\mu^2}{\sigma_\varepsilon^2\left(\sigma_\varepsilon^2+d\sigma_\mu^2\right)}\left[\left(1-\rho\right)\left(T-1\right)-\left(1-\rho^{T-1}\right)\right]$$

If we set again $\eta=\dfrac{\sigma_\mu^2}{\sigma_\varepsilon^2+d\sigma_\mu^2}$ and recall that $\left(1-\rho^2\right)\sigma_\nu^2=\sigma_\varepsilon^2$, we finally obtain

$$tr\left(\Sigma^{-1}\frac{\partial A_\rho}{\partial\rho}\right)=-\frac{4N\eta}{\left(1-\rho^2\right)\sigma_\nu^2}\left[\left(1-\rho\right)\left(T-1\right)-\left(1-\rho^{T-1}\right)\right]$$

known as equation (37).

# Appendix D: Information Matrix

We are interested in deriving the components of the information matrix.

$$J(\gamma) = -E \begin{bmatrix} \frac{\partial^2 l(\cdot)}{\partial\beta\partial\beta'} & \frac{\partial^2 l(\cdot)}{\partial\beta\partial\rho} & \frac{\partial^2 l(\cdot)}{\partial\beta\partial\sigma_\nu^2} & \frac{\partial^2 l(\cdot)}{\partial\beta\partial\sigma_\mu^2} \\ \frac{\partial^2 l(\cdot)}{\partial\beta\partial\rho} & \frac{\partial^2 l(\cdot)}{(\partial\rho)^2} & \frac{\partial^2 l(\cdot)}{\partial\rho\partial\sigma_\nu^2} & \frac{\partial^2 l(\cdot)}{\partial\rho\partial\sigma_\mu^2} \\ \frac{\partial^2 l(\cdot)}{\partial\beta\partial\sigma_\nu^2} & \frac{\partial^2 l(\cdot)}{\partial\rho\partial\sigma_\nu^2} & \frac{\partial^2 l(\cdot)}{(\partial\sigma_\nu^2)^2} & \frac{\partial^2 l(\cdot)}{\partial\sigma_\nu^2\partial\sigma_\mu^2} \\ \frac{\partial^2 l(\cdot)}{\partial\beta\partial\sigma_\mu^2} & \frac{\partial^2 l(\cdot)}{\partial\rho\partial\sigma_\mu^2} & \frac{\partial^2 l(\cdot)}{\partial\sigma_\nu^2\partial\sigma_\mu^2} & \frac{\partial^2 l(\cdot)}{\left(\partial\sigma_\mu^2\right)^2} \end{bmatrix}$$

$$J(\gamma) = \begin{bmatrix} X'\Sigma^{-1}X & 0 & 0 & 0 \\ 0 & M_1 & M_2 & M_3 \\ 0 & M_2 & M_4 & M_5 \\ 0 & M_3 & M_5 & M_6 \end{bmatrix} = \begin{pmatrix} V_\beta^{-1} & 0 \\ 0 & V_\theta^{-1} \end{pmatrix}$$

We firstly have,

$$\begin{aligned} M_1 &= -E\left[\frac{\partial^2 l(\cdot)}{(\partial\rho)^2}\right] = -\frac{1}{2}\frac{\partial}{\partial\rho}\left[tr\left(\Sigma^{-1}A_\rho\right)\right] + \frac{1}{2}tr\left(\Sigma^{-1}\frac{\partial A_\rho}{\partial\rho}\right) \\ &= \frac{N(T-1)}{\sigma_\nu^2}\left(\frac{1+\rho^2}{(1-\rho^2)^2} - \frac{2}{(1+\rho)^2}\eta + \frac{1-\rho}{1+\rho}\frac{\partial\eta}{\partial\rho}\right) \\ &\quad - \left[\frac{2N\eta}{(1-\rho^2)\sigma_\nu^2}\left[(1-\rho)(T-1) - \left(1-\rho^{T-1}\right)\right]\right] \end{aligned}$$

where

$$\begin{aligned} \frac{\partial\eta}{\partial\rho} &= \frac{\partial}{\partial\rho}\left(\frac{\sigma_\mu^2}{\sigma_\varepsilon^2 + d\sigma_\mu^2}\right) = \frac{\partial}{\partial\rho}\left(\frac{\sigma_\mu^2}{(1-\rho^2)\sigma_\nu^2 + \left(1-\rho^2+(T-1)(1-\rho)^2\right)\sigma_\mu^2}\right) \\ &= \frac{2\eta^2}{\sigma_\mu^2}\left(\rho\sigma_\nu^2 + \left(\frac{d}{1-\rho} - 1\right)\sigma_\mu^2\right) \end{aligned} \tag{D-1}$$

Hence,

$$M_1 = \frac{N(T-1)(1+\rho^2)}{(1-\rho^2)^2\sigma_\nu^2}$$

$$
\begin{aligned}
&-\frac{2N}{(1+\rho)\,\sigma_{\nu}^{2}}\left(\frac{T-1}{1+\rho}+T-1-\frac{1-\rho^{T-1}}{1-\rho}\right)\eta \\
&+\frac{2N\,(T-1)\,(1-\rho)}{(1+\rho)\,\sigma_{\mu}^{2}\sigma_{\nu}^{2}}\left(\rho\sigma_{\nu}^{2}+\left(\frac{d}{1-\rho}-1\right)\sigma_{\mu}^{2}\right)\eta^{2} \qquad \text{(D-2)}
\end{aligned}
$$

Secondly, we have

$$
\begin{aligned}
M_2 &= -E\left[\frac{\partial^2 l(\cdot)}{\partial\rho\partial\sigma_{\nu}^{2}}\right] = -\frac{1}{2}\frac{\partial}{\partial\rho}\left[tr\left(\Sigma^{-1}A\right)\right]+\frac{1}{2}tr\left(\Sigma^{-1}A_{\rho}\right) \\
&= \frac{N}{2\sigma_{\nu}^{2}}\frac{\partial}{\partial\rho}(d\eta)-\frac{N\,(T-1)}{(1-\rho^2)\,\sigma_{\nu}^{2}}\left(\rho+(1-\rho)^2\eta\right)
\end{aligned}
$$

with

$$
\begin{aligned}
\frac{\partial}{\partial\rho}(d\eta) &= d\frac{\partial\eta}{\partial\rho}+\eta\frac{\partial}{\partial\rho}\left(1-\rho^2+(T-1)\,(1-\rho)^2\right) \\
&= \frac{2d\eta^2}{\sigma_{\mu}^{2}}\left(\rho\sigma_{\nu}^{2}+\left(\frac{d}{1-\rho}-1\right)\sigma_{\mu}^{2}\right)-2\eta\left(\frac{d}{1-\rho}-1\right) \qquad \text{(D-3)}
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
M_2 &= -\frac{N\,(T-1)\,\rho}{(1-\rho^2)\,\sigma_{\nu}^{2}}-\frac{N}{\sigma_{\nu}^{2}}\left(\frac{d}{1-\rho}-1+\frac{(T-1)\,(1-\rho)}{1+\rho}\right)\eta \\
&+\frac{Nd}{\sigma_{\mu}^{2}\sigma_{\nu}^{2}}\left(\rho\sigma_{\nu}^{2}+\left(\frac{d}{1-\rho}-1\right)\sigma_{\mu}^{2}\right)\eta^2 \qquad \text{(D-4)}
\end{aligned}
$$

Thirdly, we have

$$
\begin{aligned}
M_3 &= -E\left[\frac{\partial^2 l(\cdot)}{\partial\rho\partial\sigma_{\mu}^{2}}\right] = -\frac{1}{2}\frac{\partial}{\partial\rho}\left[tr\left(\Sigma^{-1}B\right)\right] = -\frac{1}{2}\frac{\partial}{\partial\rho}\left[\frac{Nd}{\sigma_{\mu}^{2}}\eta\right] = -\frac{N}{2\sigma_{\mu}^{2}}\frac{\partial}{\partial\rho}(d\eta) \\
&= \frac{N}{\sigma_{\mu}^{2}}\left(\frac{d}{1-\rho}-1\right)\eta-\frac{Nd}{\sigma_{\mu}^{4}}\left(\rho\sigma_{\nu}^{2}+\left(\frac{d}{1-\rho}-1\right)\sigma_{\mu}^{2}\right)\eta^2 \qquad \text{(D-5)}
\end{aligned}
$$

Fourthly, we have

$$
\begin{aligned}
M_4 &= -E\left[\frac{\partial^2 l(\cdot)}{\left(\partial\sigma_{\nu}^{2}\right)^2}\right] = -\frac{1}{2}\frac{\partial}{\partial\sigma_{\nu}^{2}}\left[tr\left(\Sigma^{-1}A\right)\right] = -\frac{N}{2}\frac{\partial}{\partial\sigma_{\nu}^{2}}\left(\frac{T-d\eta}{\sigma_{\nu}^{2}}\right) \\
&= -\frac{N}{2\sigma_{\nu}^{4}}\left(-d\sigma_{\nu}^{2}\frac{\partial\eta}{\partial\sigma_{\nu}^{2}}-T+d\eta\right)
\end{aligned}
$$

with

$$\frac{\partial \eta}{\partial \sigma_{\nu}^{2}} = \frac{\partial}{\partial \sigma_{\nu}^{2}} \left( \frac{\sigma_{\mu}^{2}}{(1-\rho^2)\,\sigma_{\nu}^{2} + d\sigma_{\mu}^{2}} \right)$$

$$= -\frac{1-\rho^2}{\sigma_{\mu}^{2}} \eta^2 \tag{D-6}$$

As a consequence,

$$M_4 = \frac{NT}{2\sigma_{\nu}^{4}} - \frac{Nd}{2\sigma_{\nu}^{4}} \eta - \frac{Nd\,(1-\rho^2)}{2\sigma_{\nu}^{2}\sigma_{\mu}^{2}} \eta^2 \tag{D-7}$$

Fifthly, we get

$$M_5 = -E\left[\frac{\partial^2 l\,(\cdot)}{\partial \sigma_{\nu}^{2} \partial \sigma_{\mu}^{2}}\right] = -\frac{1}{2} \frac{\partial}{\partial \sigma_{\mu}^{2}} \left[tr\left(\Sigma^{-1} A\right)\right] = \frac{Nd}{2\sigma_{\nu}^{2}} \frac{\partial \eta}{\partial \sigma_{\mu}^{2}}$$

where

$$\frac{\partial \eta}{\partial \sigma_{\mu}^{2}} = \frac{\partial}{\partial \sigma_{\mu}^{2}} \left( \frac{\sigma_{\mu}^{2}}{\sigma_{\varepsilon}^{2} + d\sigma_{\mu}^{2}} \right) = \frac{\sigma_{\varepsilon}^{2} + d\sigma_{\mu}^{2} - d\sigma_{\mu}^{2}}{\left(\sigma_{\varepsilon}^{2} + d\sigma_{\mu}^{2}\right)^2}$$

$$= \frac{\sigma_{\varepsilon}^{2}}{\sigma_{\mu}^{4}} \eta^2 = \frac{(1-\rho^2)\,\sigma_{\nu}^{2}}{\sigma_{\mu}^{4}} \eta^2 \tag{D-8}$$

Hence,

$$M_5 = \frac{Nd\,(1-\rho^2)}{2\sigma_{\mu}^{4}} \eta^2 \tag{D-9}$$

Lastly,

$$M_6 = -E\left[\frac{\partial^2 l\,(\cdot)}{\left(\partial \sigma_{\mu}^{2}\right)^2}\right] = -\frac{1}{2} \frac{\partial}{\partial \sigma_{\mu}^{2}} \left[tr\left(\Sigma^{-1} B\right)\right] = -\frac{1}{2} \frac{\partial}{\partial \sigma_{\mu}^{2}} \left(\frac{Nd}{\sigma_{\mu}^{2}} \eta\right)$$

$$= -\frac{1}{2} \frac{\partial}{\partial \sigma_{\mu}^{2}} \left( \frac{Nd}{(1-\rho^2)\,\sigma_{\nu}^{2} + d\sigma_{\mu}^{2}} \right) = \frac{Nd}{2} \frac{d}{\left(\sigma_{\varepsilon}^{2} + d\sigma_{\mu}^{2}\right)^2}$$

$$= \frac{Nd^2}{2\sigma_{\mu}^{4}} \eta^2 \tag{D-10}$$

As a conclusion, the information matrix can be written as

$$J(\gamma) = \begin{bmatrix} X'\Sigma^{-1}X & 0 & 0 & 0 \\ 0 & M_1 & M_2 & M_3 \\ 0 & M_2 & M_4 & M_5 \\ 0 & M_3 & M_5 & M_6 \end{bmatrix} = \begin{pmatrix} V_{\beta}^{-1} & 0 \\ 0 & V_{\theta}^{-1} \end{pmatrix}$$

with

$$V_{\beta} = \left(X'\Sigma^{-1}X\right)^{-1} \text{ and } V_{\theta} = \begin{pmatrix} M_1 & M_2 & M_3 \\ M_2 & M_4 & M_5 \\ M_3 & M_5 & M_6 \end{pmatrix}^{-1}$$

where $M_i$s $i = 1, \ldots, 6$ are given above.

Table 1 : Simulation Results

| | Ordinary predictor | | | Truncated predictor | | | Misspecified predictor | | | Fixed effect predictor | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | MSE | AMSE | Z | MSE | AMSE | Z | MSE | AMSE | Z | MSE | AMSE | Z |
| **N=50. T=10** | | | | | | | | | | | | |
| $\rho = 0.01 . \phi = 0.01$ | 19.878 | 19.992 | 0.021 | 19.877 | 20.003 | 0.024 | 19.877 | 20.017 | 0.026 | 21.202 | 20.223 | -0.182 |
| $\rho = 0.01 . \phi = 0.3$ | 14.829 | 14.403 | -0.115 | 19.869 | 20.312 | 0.088 | 19.888 | 20.141 | 0.051 | 14.992 | 13.599 | -0.398 |
| $\rho = 0.01 . \phi = 0.6$ | 8.524 | 8.789 | 0.116 | 19.676 | 19.443 | -0.046 | 19.717 | 19.266 | -0.090 | 8.567 | 8.269 | -0.136 |
| $\rho = 0.01 . \phi = 0.9$ | 2.137 | 2.269 | 0.236 | 19.482 | 19.776 | 0.143 | 19.546 | 20.171 | 0.121 | 2.142 | 2.173 | 0.059 |
| $\rho = 0.3 . \phi = 0.01$ | 18.077 | 17.185 | -0.206 | 19.859 | 18.648 | -0.243 | 19.869 | 18.683 | -0.239 | 21.105 | 20.625 | -0.500 |
| $\rho = 0.3 . \phi = 0.3$ | 13.815 | 13.904 | 0.024 | 19.755 | 20.506 | 0.138 | 19.882 | 20.474 | 0.110 | 14.923 | 14.094 | -0.220 |
| $\rho = 0.3 . \phi = 0.6$ | 8.739 | 8.675 | -0.155 | 19.468 | 20.536 | 0.198 | 19.714 | 20.301 | 0.110 | 8.527 | 7.692 | -0.416 |
| $\rho = 0.3 . \phi = 0.9$ | 3.579 | 3.454 | -0.137 | 19.181 | 19.755 | 0.109 | 19.545 | 19.489 | -0.011 | 2.132 | 2.082 | -0.087 |
| $\rho = 0.6 . \phi = 0.01$ | 12.705 | 12.515 | -0.061 | 19.795 | 19.800 | 0.001 | 19.844 | 19.696 | -0.029 | 20.555 | 19.873 | -0.132 |
| $\rho = 0.6 . \phi = 0.3$ | 10.742 | 10.231 | -0.192 | 19.607 | 20.900 | 0.254 | 19.864 | 20.626 | 0.150 | 14.534 | 14.830 | 0.079 |
| $\rho = 0.6 . \phi = 0.6$ | 9.228 | 9.215 | -0.006 | 19.236 | 20.111 | 0.146 | 19.703 | 20.773 | 0.107 | 8.305 | 8.563 | 0.111 |
| $\rho = 0.6 . \phi = 0.9$ | 7.851 | 7.768 | -0.043 | 18.866 | 18.694 | -0.036 | 19.543 | 18.298 | -0.169 | 2.076 | 2.011 | -0.122 |
| $\rho = 0.9 . \phi = 0.01$ | 3.769 | 3.709 | -0.059 | 19.556 | 20.453 | 0.147 | 19.647 | 20.223 | 0.194 | 20.459 | 21.411 | 0.110 |
| $\rho = 0.9 . \phi = 0.3$ | 4.817 | 4.943 | 0.150 | 19.335 | 19.790 | 0.091 | 19.725 | 19.483 | -0.049 | 13.517 | 14.173 | 0.194 |
| $\rho = 0.9 . \phi = 0.6$ | 8.767 | 8.793 | 0.011 | 18.934 | 18.295 | -0.138 | 19.624 | 18.925 | -0.181 | 7.438 | 8.461 | 0.172 |
| $\rho = 0.9 . \phi = 0.9$ | 14.482 | 14.583 | 0.025 | 18.534 | 19.807 | 0.198 | 19.523 | 20.195 | 0.121 | 1.160 | 1.309 | 0.111 |
| **N=50. T=20** | | | | | | | | | | | | |
| $\rho = 0.01 . \phi = 0.01$ | 19.895 | 19.873 | -0.005 | 19.896 | 19.909 | 0.003 | 19.896 | 19.961 | 0.013 | 20.951 | 20.075 | -0.175 |
| $\rho = 0.01 . \phi = 0.3$ | 14.666 | 15.289 | 0.161 | 19.883 | 20.917 | 0.197 | 19.914 | 20.784 | 0.168 | 14.814 | 14.968 | 0.040 |
| $\rho = 0.01 . \phi = 0.6$ | 8.410 | 7.960 | -0.215 | 19.683 | 20.192 | 0.095 | 19.750 | 19.678 | -0.013 | 8.465 | 7.622 | -0.115 |

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\rho = 0.01 . \phi = 0.9$ | 2.107 | 1.912 | -0.402 | 19.484 | 21.957 | 0.428 | 20.587 | 21.601 | 0.154 | 2.116 | 1.920 | -0.124 |
| $\rho = 0.3 . \phi = 0.01$ | 18.096 | 17.282 | -0.168 | 19.880 | 18.444 | -0.291 | 19.892 | 18.547 | -0.269 | 20.927 | 19.544 | -0.175 |
| $\rho = 0.3 . \phi = 0.3$ | 13.732 | 13.345 | -0.107 | 19.771 | 20.356 | 0.112 | 19.911 | 20.111 | 0.039 | 14.797 | 13.961 | -0.237 |
| $\rho = 0.3 . \phi = 0.6$ | 8.672 | 8.258 | -0.200 | 19.477 | 19.456 | -0.004 | 19.749 | 18.973 | -0.153 | 8.455 | 7.530 | -0.175 |
| $\rho = 0.3 . \phi = 0.9$ | 3.560 | 3.356 | -0.244 | 19.184 | 18.443 | -0.153 | 19.587 | 18.117 | -0.307 | 2.114 | 1.982 | -0.167 |
| $\rho = 0.6 . \phi = 0.01$ | 12.719 | 12.593 | -0.039 | 19.819 | 20.452 | 0.124 | 19.870 | 20.500 | 0.123 | 20.656 | 20.508 | -0.029 |
| $\rho = 0.6 . \phi = 0.3$ | 10.820 | 10.439 | -0.138 | 19.623 | 18.168 | -0.302 | 19.896 | 18.032 | -0.386 | 14.605 | 13.075 | -0.234 |
| $\rho = 0.6 . \phi = 0.6$ | 9.299 | 9.805 | 0.195 | 19.246 | 20.584 | 0.260 | 19.740 | 20.244 | 0.099 | 8.346 | 8.058 | -0.138 |
| $\rho = 0.6 . \phi = 0.9$ | 7.872 | 8.288 | 0.199 | 19.869 | 20.377 | 0.195 | 19.585 | 19.813 | 0.045 | 2.086 | 2.043 | -0.084 |
| $\rho = 0.9 . \phi = 0.01$ | 3.771 | 3.841 | 0.068 | 19.593 | 19.311 | -0.058 | 19.681 | 19.468 | -0.043 | 18.277 | 19.110 | 0.186 |
| $\rho = 0.9 . \phi = 0.3$ | 4.989 | 5.153 | 0.126 | 19.360 | 18.854 | -0.107 | 19.762 | 18.345 | -0.308 | 12.802 | 13.075 | 0.189 |
| $\rho = 0.9 . \phi = 0.6$ | 9.037 | 8.908 | -0.053 | 18.949 | 19.248 | 0.060 | 19.664 | 18.661 | -0.205 | 6.173 | 7.961 | 0.142 |
| $\rho = 0.9 . \phi = 0.9$ | 14.593 | 14.168 | -0.120 | 18.537 | 19.512 | 0.198 | 19.565 | 19.045 | -0.107 | 1.543 | 1.858 | 0.194 |

**N=50. T=50**

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\rho = 0.01 . \phi = 0.01$ | 19.901 | 19.044 | -0.181 | 19.902 | 19.042 | -0.181 | 19.902 | 19.108 | -0.166 | 20.731 | 19.463 | -0.259 |
| $\rho = 0.01 . \phi = 0.3$ | 14.477 | 13.619 | -0.250 | 19.884 | 19.942 | 0.011 | 19.891 | 19.690 | -0.040 | 14.659 | 13.422 | -0.364 |
| $\rho = 0.01 . \phi = 0.6$ | 8.287 | 9.011 | 0.200 | 19.684 | 20.630 | 0.179 | 19.698 | 20.297 | 0.114 | 8.376 | 8.786 | 0.173 |
| $\rho = 0.01 . \phi = 0.9$ | 2.074 | 1.936 | -0.276 | 19.484 | 19.251 | -0.046 | 19.506 | 18.756 | -0.150 | 2.094 | 1.881 | -0.440 |
| $\rho = 0.3 . \phi = 0.01$ | 18.099 | 19.776 | 0.109 | 20.878 | 21.198 | 0.132 | 20.896 | 21.248 | 0.237 | 20.844 | 21.525 | 0.118 |
| $\rho = 0.3 . \phi = 0.3$ | 13.620 | 13.358 | -0.078 | 19.768 | 19.106 | -0.136 | 19.887 | 18.856 | -0.213 | 14.738 | 13.578 | -0.333 |
| $\rho = 0.3 . \phi = 0.6$ | 8.592 | 8.771 | 0.075 | 19.475 | 22.092 | 0.485 | 20.696 | 21.653 | 0.170 | 8.422 | 8.615 | 0.083 |
| $\rho = 0.3 . \phi = 0.9$ | 3.538 | 3.524 | -0.015 | 19.183 | 20.581 | 0.258 | 19.505 | 19.526 | 0.004 | 2.105 | 2.142 | 0.066 |
| $\rho = 0.6 . \phi = 0.01$ | 12.717 | 13.499 | 0.113 | 20.785 | 21.357 | 0.289 | 20.872 | 21.342 | 0.266 | 20.989 | 21.826 | 0.146 |
| $\rho = 0.6 . \phi = 0.3$ | 10.909 | 10.023 | -0.251 | 19.599 | 19.381 | -0.046 | 19.869 | 19.203 | -0.139 | 14.840 | 13.904 | -0.266 |
| $\rho = 0.6 . \phi = 0.6$ | 9.375 | 9.332 | -0.018 | 19.232 | 19.929 | 0.138 | 19.686 | 19.544 | -0.029 | 8.480 | 8.188 | -0.145 |
| $\rho = 0.6 . \phi = 0.9$ | 7.895 | 7.421 | -0.258 | 19.865 | 20.253 | 0.262 | 19.503 | 19.690 | 0.037 | 2.120 | 1.964 | -0.117 |

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\rho = 0.9 . \phi = 0.01$ | 3.768 | 3.968 | 0.202 | 19.578 | 18.254 | -0.269 | 19.636 | 18.315 | -0.265 | 18.246 | 18.718 | 0.092 |
| $\rho = 0.9 . \phi = 0.3$ | 5.276 | 5.055 | -0.167 | 19.350 | 18.301 | -0.207 | 19.702 | 18.160 | -0.308 | 12.901 | 13.609 | 0.188 |
| $\rho = 0.9 . \phi = 0.6$ | 9.445 | 9.642 | 0.082 | 18.943 | 19.597 | 0.118 | 19.591 | 20.299 | 0.267 | 7.372 | 8.515 | 0.505 |
| $\rho = 0.9 . \phi = 0.9$ | 14.749 | 14.956 | 0.055 | 18.536 | 19.958 | 0.134 | 19.479 | 20.418 | 0.175 | 1.843 | 2.108 | 0.482 |

**N=100. T=10**

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\rho = 0.01 . \phi = 0.01$ | 19.861 | 20.326 | 0.083 | 19.862 | 20.322 | 0.082 | 19.862 | 20.376 | 0.091 | 21.190 | 20.331 | -0.153 |
| $\rho = 0.01 . \phi = 0.3$ | 14.821 | 15.377 | 0.144 | 19.909 | 20.672 | 0.147 | 19.921 | 20.492 | 0.111 | 14.983 | 14.481 | -0.137 |
| $\rho = 0.01 . \phi = 0.6$ | 8.519 | 8.346 | -0.083 | 19.773 | 19.051 | -0.148 | 19.800 | 18.876 | -0.188 | 8.562 | 8.079 | -0.231 |
| $\rho = 0.01 . \phi = 0.9$ | 2.136 | 2.302 | 0.283 | 19.637 | 20.186 | 0.108 | 19.679 | 19.797 | 0.024 | 2.140 | 2.165 | 0.044 |
| $\rho = 0.3 . \phi = 0.01$ | 18.062 | 17.489 | -0.115 | 19.847 | 19.008 | -0.156 | 19.854 | 19.052 | -0.147 | 20.091 | 19.156 | -0.356 |
| $\rho = 0.3 . \phi = 0.3$ | 13.808 | 14.314 | 0.142 | 20.829 | 21.004 | 0.122 | 19.916 | 21.030 | 0.210 | 14.913 | 14.684 | -0.064 |
| $\rho = 0.3 . \phi = 0.6$ | 8.729 | 8.860 | 0.057 | 20.628 | 21.019 | 0.155 | 19.797 | 20.749 | 0.177 | 8.522 | 8.318 | -0.097 |
| $\rho = 0.3 . \phi = 0.9$ | 3.565 | 3.778 | 0.203 | 19.427 | 20.901 | 0.162 | 19.678 | 20.621 | 0.168 | 2.130 | 2.047 | -0.152 |
| $\rho = 0.6 . \phi = 0.01$ | 12.696 | 12.224 | -0.152 | 19.802 | 19.756 | -0.009 | 19.831 | 19.771 | -0.012 | 20.539 | 19.765 | -0.159 |
| $\rho = 0.6 . \phi = 0.3$ | 10.731 | 11.212 | 0.165 | 19.724 | 19.692 | -0.007 | 19.899 | 19.647 | -0.051 | 14.522 | 14.409 | -0.030 |
| $\rho = 0.6 . \phi = 0.6$ | 9.198 | 8.509 | -0.329 | 19.466 | 20.537 | 0.207 | 19.787 | 20.324 | 0.103 | 8.298 | 7.809 | -0.250 |
| $\rho = 0.6 . \phi = 0.9$ | 7.797 | 7.611 | -0.092 | 19.206 | 20.428 | 0.236 | 19.675 | 20.175 | 0.097 | 2.075 | 1.928 | -0.302 |
| $\rho = 0.9 . \phi = 0.01$ | 3.767 | 3.815 | 0.049 | 19.635 | 19.334 | -0.060 | 19.685 | 19.020 | -0.135 | 18.455 | 19.437 | 0.165 |
| $\rho = 0.9 . \phi = 0.3$ | 4.802 | 4.828 | 0.020 | 19.535 | 19.984 | 0.085 | 19.796 | 19.835 | 0.007 | 12.513 | 13.888 | 0.158 |
| $\rho = 0.9 . \phi = 0.6$ | 8.709 | 9.247 | 0.233 | 19.254 | 20.433 | 0.220 | 19.729 | 20.208 | 0.090 | 7.436 | 8.283 | 0.186 |
| $\rho = 0.9 . \phi = 0.9$ | 14.366 | 14.263 | -0.028 | 19.974 | 20.241 | 0.249 | 19.661 | 19.926 | 0.053 | 1.959 | 2.080 | 0.034 |

**N=100. T=20**

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\rho = 0.01 . \phi = 0.01$ | 20.861 | 21.001 | 0.219 | 19.862 | 20.995 | 0.218 | 19.862 | 20.945 | 0.208 | 20.891 | 21.112 | 0.042 |
| $\rho = 0.01 . \phi = 0.3$ | 14.646 | 13.911 | -0.198 | 19.910 | 19.420 | -0.100 | 19.932 | 19.437 | -0.100 | 14.771 | 13.619 | -0.320 |

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\rho = 0.01 . \phi = 0.6$ | 8.399 | 8.562 | 0.074 | 19.774 | 19.181 | -0.123 | 19.822 | 19.076 | -0.154 | 8.441 | 8.110 | -0.160 |
| $\rho = 0.01 . \phi = 0.9$ | 2.104 | 2.144 | 0.073 | 19.638 | 19.989 | 0.068 | 19.711 | 19.760 | 0.010 | 2.110 | 2.072 | -0.072 |
| $\rho = 0.3 . \phi = 0.01$ | 18.064 | 17.468 | -0.137 | 19.851 | 19.119 | -0.152 | 19.858 | 19.170 | -0.144 | 20.852 | 19.241 | -0.339 |
| $\rho = 0.3 . \phi = 0.3$ | 13.713 | 13.390 | -0.096 | 19.832 | 19.531 | -0.062 | 19.929 | 19.524 | -0.083 | 14.744 | 14.069 | -0.188 |
| $\rho = 0.3 . \phi = 0.6$ | 8.655 | 9.354 | 0.289 | 19.630 | 20.361 | 0.133 | 19.820 | 20.171 | 0.065 | 8.425 | 8.925 | 0.217 |
| $\rho = 0.3 . \phi = 0.9$ | 3.544 | 3.364 | -0.197 | 19.427 | 20.405 | 0.178 | 19.711 | 19.939 | 0.043 | 2.106 | 2.071 | -0.065 |
| $\rho = 0.6 . \phi = 0.01$ | 12.698 | 12.393 | -0.097 | 19.811 | 19.426 | -0.073 | 19.842 | 19.491 | -0.066 | 20.560 | 19.450 | -0.209 |
| $\rho = 0.6 . \phi = 0.3$ | 10.801 | 10.913 | 0.039 | 19.731 | 20.274 | 0.106 | 19.918 | 20.086 | 0.033 | 14.538 | 14.422 | -0.031 |
| $\rho = 0.6 . \phi = 0.6$ | 9.264 | 8.834 | -0.189 | 19.469 | 19.552 | 0.016 | 19.814 | 19.347 | -0.091 | 8.307 | 7.705 | -0.309 |
| $\rho = 0.6 . \phi = 0.9$ | 7.818 | 7.800 | -0.009 | 19.207 | 19.937 | 0.144 | 19.709 | 19.750 | 0.008 | 2.077 | 2.055 | -0.040 |
| $\rho = 0.9 . \phi = 0.01$ | 3.767 | 3.771 | 0.004 | 20.661 | 21.119 | 0.277 | 19.730 | 20.921 | 0.229 | 20.214 | 21.088 | 0.095 |
| $\rho = 0.9 . \phi = 0.3$ | 4.973 | 4.863 | -0.089 | 19.552 | 19.168 | -0.076 | 19.839 | 18.966 | -0.174 | 12.757 | 13.428 | 0.185 |
| $\rho = 0.9 . \phi = 0.6$ | 8.978 | 8.812 | -0.074 | 19.264 | 19.372 | 0.022 | 19.768 | 19.205 | -0.113 | 7.147 | 8.227 | 0.126 |
| $\rho = 0.9 . \phi = 0.9$ | 14.476 | 15.172 | 0.171 | 19.976 | 20.808 | 0.234 | 19.698 | 20.420 | 0.137 | 1.537 | 1.917 | 0.159 |

**N=100. T=50**

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\rho = 0.01 . \phi = 0.01$ | 19.875 | 20.317 | 0.090 | 19.877 | 20.329 | 0.092 | 19.877 | 20.360 | 0.099 | 20.649 | 20.379 | -0.055 |
| $\rho = 0.01 . \phi = 0.3$ | 14.463 | 14.267 | -0.052 | 19.919 | 19.594 | -0.065 | 19.925 | 19.607 | -0.064 | 14.600 | 13.919 | -0.183 |
| $\rho = 0.01 . \phi = 0.6$ | 8.279 | 8.440 | 0.077 | 19.779 | 20.515 | 0.139 | 19.792 | 20.358 | 0.108 | 8.343 | 8.210 | -0.064 |
| $\rho = 0.01 . \phi = 0.9$ | 2.073 | 2.093 | 0.038 | 19.639 | 19.577 | -0.013 | 19.659 | 19.261 | -0.085 | 2.086 | 2.107 | 0.039 |
| $\rho = 0.3 . \phi = 0.01$ | 18.077 | 19.410 | 0.261 | 20.861 | 21.219 | 0.242 | 20.873 | 21.191 | 0.137 | 20.719 | 21.396 | 0.121 |
| $\rho = 0.3 . \phi = 0.3$ | 13.607 | 13.040 | -0.176 | 19.838 | 20.814 | 0.184 | 19.922 | 20.704 | 0.146 | 14.650 | 14.101 | -0.157 |
| $\rho = 0.3 . \phi = 0.6$ | 8.578 | 8.602 | 0.011 | 19.633 | 20.573 | 0.181 | 19.790 | 20.431 | 0.125 | 8.371 | 8.080 | -0.140 |
| $\rho = 0.3 . \phi = 0.9$ | 3.523 | 3.813 | 0.290 | 19.428 | 20.859 | 0.269 | 19.658 | 20.461 | 0.154 | 2.093 | 2.091 | -0.003 |
| $\rho = 0.6 . \phi = 0.01$ | 12.704 | 12.825 | 0.037 | 19.797 | 19.458 | -0.070 | 19.856 | 19.819 | -0.007 | 20.776 | 20.211 | -0.114 |
| $\rho = 0.6 . \phi = 0.3$ | 10.893 | 10.723 | -0.059 | 19.720 | 19.945 | 0.044 | 19.910 | 19.854 | -0.011 | 14.690 | 13.731 | -0.268 |
| $\rho = 0.6 . \phi = 0.6$ | 9.341 | 9.119 | -0.094 | 19.463 | 19.867 | 0.077 | 19.783 | 19.700 | -0.016 | 8.394 | 8.273 | -0.055 |

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\rho = 0.6 . \phi = 0.9$ | 7.840 | 7.149 | -0.372 | 19.206 | 19.239 | 0.007 | 19.656 | 18.938 | -0.154 | 2.099 | 2.149 | 0.092 |
| $\rho = 0.9 . \phi = 0.01$ | 3.766 | 3.797 | 0.029 | 19.653 | 19.960 | 0.059 | 19.690 | 19.530 | -0.032 | 19.908 | 20.335 | 0.170 |
| $\rho = 0.9 . \phi = 0.3$ | 5.258 | 5.323 | 0.050 | 19.547 | 19.767 | 0.043 | 19.793 | 19.494 | -0.058 | 13.662 | 14.595 | 0.109 |
| $\rho = 0.9 . \phi = 0.6$ | 9.382 | 8.827 | -0.231 | 19.261 | 19.339 | 0.016 | 19.716 | 19.167 | -0.112 | 7.235 | 8.239 | 0.166 |
| $\rho = 0.9 . \phi = 0.9$ | 14.630 | 15.276 | 0.162 | 20.975 | 21.046 | 0.184 | 19.640 | 20.511 | 0.168 | 1.809 | 2.032 | 0.123 |

**N=500. T=10**

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\rho = 0.01 . \phi = 0.01$ | 19.873 | 20.885 | 0.183 | 19.874 | 20.882 | 0.183 | 19.874 | 20.860 | 0.178 | 21.015 | 20.822 | -0.116 |
| $\rho = 0.01 . \phi = 0.3$ | 15.152 | 14.720 | -0.111 | 19.938 | 19.929 | -0.002 | 19.945 | 19.810 | -0.027 | 15.566 | 14.021 | -0.118 |
| $\rho = 0.01 . \phi = 0.6$ | 8.761 | 8.325 | -0.206 | 19.819 | 19.389 | -0.087 | 19.836 | 19.202 | -0.129 | 8.895 | 7.831 | -0.127 |
| $\rho = 0.01 . \phi = 0.9$ | 2.202 | 2.184 | -0.031 | 20.700 | 21.595 | 0.144 | 19.727 | 21.165 | 0.267 | 2.224 | 1.966 | -0.188 |
| $\rho = 0.3 . \phi = 0.01$ | 18.072 | 17.561 | -0.116 | 19.857 | 19.364 | -0.103 | 19.865 | 19.369 | -0.103 | 20.814 | 19.509 | -0.177 |
| $\rho = 0.3 . \phi = 0.3$ | 13.980 | 13.811 | -0.047 | 19.867 | 19.027 | -0.173 | 19.939 | 19.067 | -0.180 | 14.424 | 13.928 | -0.128 |
| $\rho = 0.3 . \phi = 0.6$ | 8.870 | 9.463 | 0.244 | 19.695 | 19.791 | 0.019 | 19.832 | 19.569 | -0.053 | 8.814 | 8.302 | -0.233 |
| $\rho = 0.3 . \phi = 0.9$ | 3.601 | 3.599 | -0.003 | 19.524 | 18.830 | -0.150 | 19.726 | 18.562 | -0.255 | 2.203 | 1.984 | -0.138 |
| $\rho = 0.6 . \phi = 0.01$ | 12.701 | 12.634 | -0.020 | 19.806 | 20.368 | 0.107 | 19.838 | 20.282 | 0.085 | 20.622 | 20.335 | -0.054 |
| $\rho = 0.6 . \phi = 0.3$ | 10.634 | 9.930 | -0.288 | 19.770 | 18.800 | -0.212 | 19.920 | 18.827 | -0.237 | 14.581 | 13.326 | -0.358 |
| $\rho = 0.6 . \phi = 0.6$ | 9.084 | 9.118 | 0.014 | 19.554 | 20.064 | 0.099 | 19.821 | 19.924 | 0.020 | 8.332 | 8.386 | 0.026 |
| $\rho = 0.6 . \phi = 0.9$ | 7.743 | 8.449 | 0.329 | 21.338 | 22.256 | 0.177 | 20.723 | 21.990 | 0.377 | 2.083 | 2.065 | -0.035 |
| $\rho = 0.9 . \phi = 0.01$ | 3.767 | 3.889 | 0.123 | 19.641 | 19.553 | -0.018 | 19.692 | 19.436 | -0.052 | 18.130 | 19.643 | 0.157 |
| $\rho = 0.9 . \phi = 0.3$ | 4.603 | 4.640 | 0.032 | 19.594 | 19.113 | -0.099 | 19.817 | 18.934 | -0.183 | 13.869 | 14.049 | 0.169 |
| $\rho = 0.9 . \phi = 0.6$ | 8.354 | 7.943 | -0.200 | 19.368 | 19.919 | 0.111 | 19.763 | 19.511 | -0.053 | 7.497 | 8.099 | 0.163 |
| $\rho = 0.9 . \phi = 0.9$ | 14.174 | 13.547 | -0.173 | 19.142 | 19.593 | 0.086 | 19.708 | 19.294 | -0.081 | 1.924 | 2.061 | 0.170 |

**N=500. T=20**

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\rho = 0.01 . \phi = 0.01$ | 19.888 | 18.857 | -0.216 | 19.890 | 18.862 | -0.216 | 19.890 | 19.011 | -0.184 | 21.048 | 20.191 | -0.186 |

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\rho = 0.01 . \phi = 0.3$ | 14.665 | 14.769 | 0.028 | 19.949 | 21.274 | 0.243 | 20.962 | 21.285 | 0.244 | 14.882 | 14.418 | -0.128 |
| $\rho = 0.01 . \phi = 0.6$ | 8.410 | 8.451 | 0.019 | 19.825 | 20.345 | 0.101 | 19.853 | 20.293 | 0.086 | 8.504 | 8.119 | -0.185 |
| $\rho = 0.01 . \phi = 0.9$ | 2.107 | 2.117 | 0.019 | 19.701 | 20.261 | 0.106 | 19.744 | 19.888 | 0.028 | 2.126 | 2.042 | -0.163 |
| $\rho = 0.3 . \phi = 0.01$ | 18.086 | 18.822 | 0.157 | 19.869 | 20.379 | 0.097 | 19.881 | 20.406 | 0.100 | 21.096 | 20.682 | -0.078 |
| $\rho = 0.3 . \phi = 0.3$ | 13.727 | 13.326 | -0.112 | 19.876 | 20.449 | 0.107 | 19.955 | 20.370 | 0.078 | 14.916 | 14.023 | -0.249 |
| $\rho = 0.3 . \phi = 0.6$ | 8.660 | 8.652 | -0.004 | 19.700 | 19.109 | -0.119 | 19.849 | 19.062 | -0.158 | 8.524 | 8.143 | -0.179 |
| $\rho = 0.3 . \phi = 0.9$ | 3.541 | 3.643 | 0.105 | 19.525 | 20.215 | 0.135 | 19.743 | 19.880 | 0.027 | 2.131 | 1.954 | -0.345 |
| $\rho = 0.6 . \phi = 0.01$ | 12.707 | 12.228 | -0.148 | 19.806 | 18.449 | -0.287 | 19.848 | 18.489 | -0.287 | 20.960 | 19.757 | -0.160 |
| $\rho = 0.6 . \phi = 0.3$ | 10.804 | 10.174 | -0.242 | 19.770 | 20.998 | 0.226 | 19.932 | 20.991 | 0.197 | 14.820 | 14.742 | -0.021 |
| $\rho = 0.6 . \phi = 0.6$ | 9.256 | 8.974 | -0.125 | 19.554 | 18.881 | -0.133 | 19.836 | 18.632 | -0.238 | 8.468 | 8.112 | -0.179 |
| $\rho = 0.6 . \phi = 0.9$ | 7.798 | 7.878 | 0.039 | 19.339 | 20.354 | 0.192 | 19.739 | 20.223 | 0.093 | 2.117 | 2.076 | -0.077 |
| $\rho = 0.9 . \phi = 0.01$ | 3.767 | 3.841 | 0.075 | 19.674 | 19.915 | 0.046 | 19.678 | 19.737 | 0.011 | 18.649 | 19.917 | 0.106 |
| $\rho = 0.9 . \phi = 0.3$ | 4.966 | 4.837 | -0.103 | 19.617 | 19.392 | -0.043 | 19.812 | 19.079 | -0.142 | 12.065 | 13.478 | 0.157 |
| $\rho = 0.9 . \phi = 0.6$ | 8.955 | 8.574 | -0.175 | 19.381 | 20.460 | 0.207 | 19.767 | 20.159 | 0.076 | 7.323 | 8.246 | 0.130 |
| $\rho = 0.9 . \phi = 0.9$ | 14.430 | 14.391 | -0.010 | 19.145 | 20.801 | 0.298 | 19.722 | 20.450 | 0.135 | 1.581 | 1.942 | 0.126 |
| **N=500. T=50** | | | | | | | | | | | | |
| $\rho = 0.01 . \phi = 0.01$ | 19.865 | 20.348 | 0.090 | 19.868 | 20.354 | 0.090 | 19.868 | 20.400 | 0.099 | 20.617 | 20.562 | -0.010 |
| $\rho = 0.01 . \phi = 0.3$ | 14.458 | 14.206 | -0.068 | 19.932 | 19.843 | -0.017 | 19.937 | 19.798 | -0.027 | 14.578 | 14.008 | -0.157 |
| $\rho = 0.01 . \phi = 0.6$ | 8.276 | 8.910 | 0.291 | 19.815 | 20.707 | 0.173 | 19.825 | 20.608 | 0.152 | 8.330 | 8.718 | 0.181 |
| $\rho = 0.01 . \phi = 0.9$ | 2.072 | 2.229 | 0.277 | 19.699 | 19.430 | -0.053 | 19.713 | 19.171 | -0.109 | 2.083 | 2.141 | 0.108 |
| $\rho = 0.3 . \phi = 0.01$ | 18.069 | 18.074 | 0.001 | 19.854 | 20.051 | 0.039 | 19.864 | 20.048 | 0.036 | 20.671 | 20.115 | -0.108 |
| $\rho = 0.3 . \phi = 0.3$ | 13.602 | 14.577 | 0.258 | 19.865 | 20.368 | 0.094 | 19.934 | 20.275 | 0.064 | 14.616 | 15.164 | 0.139 |
| $\rho = 0.3 . \phi = 0.6$ | 8.572 | 8.109 | -0.223 | 19.694 | 20.472 | 0.149 | 19.824 | 20.393 | 0.109 | 8.352 | 7.882 | -0.232 |
| $\rho = 0.3 . \phi = 0.9$ | 3.517 | 3.448 | -0.075 | 19.524 | 19.636 | 0.023 | 19.713 | 19.471 | -0.050 | 2.088 | 2.040 | -0.086 |
| $\rho = 0.6 . \phi = 0.01$ | 12.700 | 13.151 | 0.124 | 19.802 | 20.319 | 0.097 | 19.850 | 20.415 | 0.104 | 20.696 | 20.525 | -0.031 |
| $\rho = 0.6 . \phi = 0.3$ | 10.887 | 10.914 | 0.010 | 19.768 | 20.047 | 0.054 | 19.924 | 20.001 | 0.015 | 14.633 | 14.244 | -0.108 |

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\rho = 0.6 . \phi = 0.6$ | 9.328 | 9.100 | -0.092 | 19.553 | 19.657 | 0.021 | 19.818 | 19.460 | -0.072 | 8.362 | 7.991 | -0.183 |
| $\rho = 0.6 . \phi = 0.9$ | 7.819 | 7.961 | 0.067 | 19.338 | 21.657 | 0.403 | 19.711 | 21.319 | 0.283 | 2.090 | 2.118 | 0.049 |
| $\rho = 0.9 . \phi = 0.01$ | 3.766 | 3.826 | 0.061 | 19.684 | 19.928 | 0.048 | 19.711 | 19.896 | 0.036 | 19.782 | 20.437 | 0.105 |
| $\rho = 0.9 . \phi = 0.3$ | 5.251 | 5.077 | -0.135 | 19.624 | 20.670 | 0.206 | 19.826 | 20.525 | 0.138 | 13.573 | 14.678 | 0.156 |
| $\rho = 0.9 . \phi = 0.6$ | 9.358 | 9.234 | -0.055 | 19.385 | 20.333 | 0.182 | 19.762 | 20.091 | 0.065 | 7.185 | 8.203 | 0.187 |
| $\rho = 0.9 . \phi = 0.9$ | 14.584 | 14.824 | 0.062 | 19.146 | 20.570 | 0.260 | 19.697 | 20.278 | 0.109 | 1.796 | 1.965 | 0.130 |