Testing for stationarity with a break in heterogeneous panels where the time dimension is finite

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Abstract

In this paper, we consider the case of finite time dimension in the panel stationarity tests with a structural break. By fixing T, the finite sample properties of the tests for both micro (T small and N large) and macro (T and N large) panel data are generally greatly improved. More importantly, the derivation of the tests for finite T and $N \longrightarrow \infty$, as opposed to joint asymptotics where N and $T \rightarrow \infty$ simultaneously, avoids the imposition of the rate condition $N/T \rightarrow 0,$ making the test valid for any (T, N) blend. Four models corresponding to the usual combination of breaks are considered. The asymptotic distributions of the tests are derived under the null and are shown to be normally distributed. Their moments for T fixed are derived analytically employing two approaches. The first method is based on the Laplace Transform and the second derivation is based on Ghazal's (1994) corollary 1. The case with unknown break is also considered. The proposed tests have generally empirical sizes that are very close to the nominal size. The Monte-Carlo simulations show that the power of the test statistics increases substantially with N and T.

Keywords: Panel data; Structural breaks; Stationarity tests; Moments of the ratio of two dependent quadratic forms, Laplace transform.
JEL classification: C12; C14;C23; C52.

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1 Introduction

An upsurge of interest in nonstationary panel data models has been witnessed in both theoretical and empirical research in recent years. Since the seminal papers by Breitung and Meyer (1994), Quah (1994), Maddala and Wu (1999), Phillips and Moon (1999), Levin, Lin and Chu (2002), Im, Pesaran and Shin (2003), Hadri (2000) and Hadri and Larsson (2005), panel unit root and stationarity tests, have been applied to a variety of key economic issues with the hope that the increased power of these tests, due to the exploitation of the cross-section dimension, would provide more compelling evidence. Banerjee (1999), Baltagi and Kao (2000), Baltagi (2001) and Breitung and Pesaran (2005) provide comprehensive surveys on the subject.

Additionally, since the pioneering work of Perron (1989) which illustrates the need to allow for a structural break when testing for a unit root in economic time series, the problem of structural breaks in the level/slope of a series has proved to be of considerable interest in the unit root testing literature. Perron (1989) and Amsler and Lee (1995) have found that unit root tests are biased toward accepting the false unit root null hypothesis in the presence of a structural break. It is widely accepted that the failure of taking into account structural breaks is likely to lead to a significant loss of power in unit root tests and size distortion in stationarity test. Kurozumi (2002), Lee and Strazicich (2001) and Busetti and Harvey (2001, 2003) have considered testing the null hypothesis of stationarity in the presence of a single break versus the alternative of a unit root in time series.

In the panel context, some attention has been paid, recently, to the presence of structural changes in unit root tests and stationarity tests. For example, Im, Lee and Tieslau (2005) proposed unit root panel tests, whereas Carrion-i-Silvestre, Del Barrio and López-Bazo (2005) (CBL thereafter) and Hadri and Rao (2007) developed stationarity panel tests.

Hadri and Rao (2007) proposed a test statistic with the null hypothesis of stationarity with a break against the alternative of a unit root. The asymptotic distributions of the tests are derived under the null and are shown to be normally distributed. The asymptotic distributions are derived using sequential limits, wherein $T \to \infty$ followed by $N \to \infty$. Under the rate condition $N/T \to 0$, it can be shown following Phillips and Moon (1999) that the sequential results obtained imply joint convergence. In the joint limit theory, T and N are allowed to pass to infinity simultaneously. The drawback of sequential limits is that in certain cases, they can give asymptotic results which are misleading. This is not the case for joint limit. As noted by Phillips and Moon (1999), the rate condition indicates that the joint limit theory is going to be applicable to cases when N is moderate and T is allowed to be large. This is applicable for the empirical panel data with large number of cross sections (N) and time series observations (T). However, for micro panels where T is relatively small, the tests developed under the assumption of large T are usually unsatisfactory because of size distortions. Besides, the rate condition $N/T \to 0$ limits the application of the tests to cases where T is larger than N.

In this paper, we consider the limit theory that T is fixed and N is allowed to go to infinity and we apply it to the test proposed by Hadri and Rao (2007) for testing the null of stationarity with structural breaks against the alternative of unit root in panel data. The exact expressions of the two first moments of the statistics are derived using two methods: Ghazal's (1994) corollary 1 and Laplace transform. Under this limit theory, the rate condition N/T is no longer required, which makes the test valid for any (T, N) blend and hence allows us to exploit any available number of cross sections. Moreover, the assumption that T is fixed allows the tests to be applicable to numerous empirical studies in which a large number of cross-sectional units are observed during a relatively short period of time. It is worthwhile to note that our tests are also applicable to macro panels and have better finite sample properties than tests that have been derived under the assumption that both N and T go to infinity as shown by the simulation results. We show that the asymptotic distributions of the test are normal and that their convergence rate is $\sqrt{N}$. The normality results do not require T to grow large.

The paper is organized as follows. Section 2 set up the model and assumptions. Section 3 describes the statistics and limiting distributions. We also deal with the case that the break is unknown in Section 4. Section 5 investigates the finite sample size and power of the proposed statistics via Monte Carlo simulations. Section 6 concludes the paper. All the proofs are presented in the Appendix.

2 Models and assumptions

Let us consider the following models

$$\text{Model 0:} \quad y_{it} = \alpha_i + \delta_i D_{it} + r_{it} + \epsilon_{it}, \tag{1}$$

$$\text{Model 1:} \quad y_{it} = \alpha_i + \delta_i D_{it} + \theta_i t + r_{it} + \epsilon_{it}, \tag{2}$$

$$\text{Model 2:} \quad y_{it} = \alpha_i + \theta_i t + \gamma_i DT_{it} + r_{it} + \epsilon_{it}, \tag{3}$$

$$\text{Model 3:} \quad y_{it} = \alpha_i + \delta_i D_{it} + \theta_i t + \gamma_i DT_{it} + r_{it} + \epsilon_{it}, \tag{4}$$

where y_{it}, $i = 1, ..., N$ and $t = 1, ..., T$ are the observed series for which we wish to test stationarity. For all i, $\alpha_i's$, $\theta\ _i's$, $\delta_i's$ and $\gamma_i's$ are unknown parameters. r_{it} is a random walk defined as $r_{it} = r_{it-1} + u_{it}$ with initial values $r_{i0} = 0\ \forall i$ without loss of generality (See Abadir (1993) and Abadir and Hadri (2000) for the importance of initial values in autoregressive models). Hence, under the null, the random walk vanishes from the above equations. More compactly, the four models can be expressed as follows:

$$y_{it} = z_{it}'\beta_i + r_{it} + \epsilon_{it}, \tag{5}$$

where $z_{it} = [1,\ D_{it}]'$ for Model 0 (a constant with a break); $z_{it} = [1,\ D_{it},\ t/T]'$ for model 1 (a constant with a break and a linear trend); $z_{it} = [1,\ t/T,\ DT_{it}]'$

for model 2 (a constant with no break and a linear trend with a break); $z_{it} = [1, D_{it}, t/T, DT_{it}]'$ for model 3 (a constant and a linear trend, both with a break); finally, β_i is the corresponding vector of parameters.

Assumption 2.1. *The* u_{it} and ϵ_{it} *are iid across* i and over t *with* $E(u_{it}) = 0, E(u_{it})^2 = \sigma^2_{u,i} \geq 0$. $E(\epsilon_{it}) = 0$ and $E(\epsilon_{it})^2 = \sigma^2_{\epsilon,i} > 0$.

Assumption 2.2. *Suppose that there is a one-time change in the structure that occurred at time* $T_{B,i}$*, where* $T_{B,i} = \omega_i T$, $\omega_i \in (0,1)$ *denotes the fraction of the break point to the sample for the ith individual. The dummy variables* D_{it} *and* DT_{it} *are given by* $D_{it} = 1$ *if* $t > T_{B,i}$*, and* 0 *otherwise;* $DT_{it} = t - T_{B,i}$ *if* $t > T_{B,i}$*, and* 0 *otherwise.*

Model 0, without a linear trend, describes for instance an interest rate, whereas Models 1-3 which include a linear trend, are more suitable for macroeconomic variables such as gross national product. Perron (1989) called Model 1 the "crash model" and model 2 the "changing growth model".

The null hypothesis is given by

$$H_0\colon \sigma^2_{u,i} = 0 \ \forall i, \tag{6}$$

which is tested against the alternative

$$H_1\colon \sigma^2_{u,i} > 0, i = 1, 2, ...N_1; \ \ \sigma^2_{u,i} = 0, \ \ i = N_1 + 1, ..., N. \tag{7}$$

This alternative hypothesis allows for $\sigma^2_{u,i}$ to be heterogeneous across units and includes the homogeneous alternative, i.e., $\sigma^2_{u,i} = \sigma^2_u > 0$ for all i. It also permits some of the individual series to be stationary under the alternative. The consistency of the present panel stationarity tests is guaranteed as shown by Hadri and Larsson (2005) if the fraction of the individual processes possessing a unit root is different from zero under the alternative.

3 Test statistics and limiting distributions

The statistics we consider is given by (See Hadri and Rao (2007))

$$\eta_{i,T,k}(\omega_i) = \frac{\sum_{t=1}^{T} S^2_{it}}{T^2 \widehat{\sigma}^2_{\epsilon,i}}, \tag{8}$$

with $k = 0, 1, 2, 3$ indicates the statistics corresponding to the four models. The ω_i denotes that the statistic has been constructed for a specific value of the break point location and this value is allowed to be different across individuals. $S_{it} = \sum_{j=1}^{t} \widehat{\epsilon}_{ij}$ is the partial sum process. Under the null, $\widehat{\epsilon}_{ij}$ are OLS residuals from regressing y_{it} on the appropriate set of regressors from each model and $\widehat{\sigma}^2_{\epsilon,i}$ is a consistent estimator of the variance of ϵ_{it} where

$$\widehat{\sigma}^2_{\epsilon,i} = \frac{1}{T} \sum_{t=1}^{T} \widehat{\epsilon}^2_{it}. \tag{9}$$

The Model in (5) can be expressed as a vectorized model by stacking each variable along the time dimension:

$$y_i = Z_i \beta_i + L\boldsymbol{u}_i + \boldsymbol{\epsilon}_i, \tag{10}$$

where $Z_i = [z_{i1}, z_{i2}, \ldots, z_{iT}]'$, $\boldsymbol{u}_i = [u_{i1}, u_{i2}, \ldots, u_{iT}]'$ and L is a lower triangular matrix with lower elements 1's.

The statistic in (8) can be written in matrix form as follows:

$$\eta_{i,T,k}(\omega_i) = \frac{y_i' M_i L L' M_i y_i}{T^2 \widehat{\sigma}^2_{\epsilon,i}} = T^{-1} \frac{y_i' M_i L L' M_i y_i}{y_i' M_i y_i}, \tag{11}$$

where $M_i = I_T - Z_i (Z_i' Z_i)^{-1} Z_i'$ is an idempotent matrix.

It turns out that we are able to find the exact formulae for the mean and the variance of the statistic $\eta_{i,T,k}(\omega_i)$ which are $\xi_{i,k}$ and $\varsigma^2_{i,k}$ respectively. They are given in the following theorem.

Theorem 1. *Let $\{y_{i,t}\}$ be given by* $(1) - (4)$ *Under Assumption 2.1, 2.2 for the ith individual and for a fixed sample of size T, we have,*

Model 0

$$\xi_{i,0} = \frac{2T^2\omega_i^2 - 2T^2\omega_i + T^2 - 2}{6T\,(T-2)},$$

$$\begin{aligned} \varsigma^2_{i,0} \;\; = \;\; & \frac{2\omega_i^4(T^5 - 7T^4) - 4\omega_i^3(T^5 - 7T^4) + 6\omega_i^2(T^5 - \frac{16}{3}T^4 + \frac{5}{6}T^3 + \frac{5}{3}T^2)}{45T^3\,(T-2)^2} \\ & - \frac{4\omega_i(T^5 - \frac{9}{2}T^4 + \frac{5}{4}T^3 + \frac{5}{2}T^2) + T^5 - \frac{9}{2}T^4 + \frac{5}{2}T^3 + 5T^2 + 4 - 7T}{45T^3\,(T-2)^2}; \end{aligned}$$

Model 1

$$\xi_{i,1} = \frac{15\omega_i^4 \times T^4 - 30\omega_i^3 \times T^4 + 25\omega_i^2(T^4 - 8T^2/5) - 10\omega_i(T^4 - 4T^2) + 2T^4 - 15T^2 + 13}{30T\,(T-3)\,(3T^2\omega_i^2 - 3T^2\omega_i + T^2 - 1)},$$

$$\begin{aligned} \varsigma^2_{i,1} = & \frac{315\omega_i^8(T^9 - 13T^8) - 1260\omega_i^7(T^9 - 13T^8) + 2415\omega_i^6(T^9 - \frac{289}{23}T^8 + \frac{31}{23}T^7 + \frac{67}{23}T^6)}{6300T^2(T-1)(T-3)^2\left(3T^2\omega_i^2 - 3T^2\omega_i + T^2 - 1\right)^2} \\ & - \frac{2835\omega_i^5(T^9 - \frac{107}{9}T^8 + \frac{31}{9}T^7 + \frac{67}{9}T^6)}{6300T^2(T-1)(T-3)^2\left(3T^2\omega_i^2 - 3T^2\omega_i + T^2 - 1\right)^2} \\ & + \frac{2275\omega_i^4(T^9 - \frac{709}{65}T^8 + \frac{359}{65}T^7 + \frac{863}{65}T^6 - \frac{343}{65}T^5 + \frac{233}{65}T^4)}{6300T^2(T-1)(T-3)^2\left(3T^2\omega_i^2 - 3T^2\omega_i + T^2 - 1\right)^2} \\ & - \frac{1295\omega_i^3(T^9 - \frac{359}{37}T^8 + \frac{253}{37}T^7 + \frac{721}{37}T^6 - \frac{686}{37}T^5 + \frac{466}{37}T^4)}{6300T^2(T-1)(T-3)^2\left(3T^2\omega_i^2 - 3T^2\omega_i + T^2 - 1\right)^2} \\ & + \frac{495\omega_i^2(T^9 - \frac{857}{99}T^8 + \frac{749}{99}T^7 + \frac{77}{3}T^6 - \frac{371}{9}T^5 + \frac{287}{11}T^4 + \frac{1469}{99}T^3 - \frac{1495}{99}T^2)}{6300T^2(T-1)(T-3)^2\left(3T^2\omega_i^2 - 3T^2\omega_i + T^2 - 1\right)^2} \\ & - \frac{110\omega_i(T^9 - \frac{89}{11}T^8 + \frac{189}{22}T^7 + \frac{721}{22}T^6 - \frac{840}{11}T^5 + \frac{476}{11}T^4 + \frac{1469}{22}T^3 - \frac{1495}{22}T^2)}{6300T^2(T-1)(T-3)^2\left(3T^2\omega_i^2 - 3T^2\omega_i + T^2 - 1\right)^2} \\ & + \frac{11(T^9 - \frac{89}{11}T^8 + \frac{115}{11}T^7 + 45T^6 - \frac{1477}{11}T^5 + \frac{553}{11}T^4 + \frac{2565}{11}T^3 - \frac{2235}{11}T^2 - \frac{1214}{11}T + 116)}{6300T^2(T-1)(T-3)^2\left(3T^2\omega_i^2 - 3T^2\omega_i + T^2 - 1\right)^2}; \end{aligned}$$

Model 2

$$
\begin{aligned}
\xi_{i,2} \quad = \quad & \frac{\omega_i(11T^2+5T^3-2T^4-14T)+\omega_i^2(5T^4-11T^2-9T^3)}{15T\left(T-3\right)\left(-T-2T^2\omega_i+2T^2\omega_i^2+2T\omega_i-1\right)} \\
& +\frac{\omega_i^3(6T^3-6T^4)+3T^4\omega_i^4+7T-T^2-T^3+7}{15T\left(T-3\right)\left(-T-2T^2\omega_i+2T^2\omega_i^2+2T\omega_i-1\right)},
\end{aligned}
$$

$$
\begin{aligned}
\varsigma_{i,2}^2 = & \frac{1156+1012T-1071T^2-417T^3+558T^4-30T^5-67T^6+11T^7}{6300T^2(T-1)(T-3)^2\left(-T-2T^2\omega_i+2T^2\omega_i^2+2T\omega_i-1\right)^2} \\
& +\frac{\omega_i\left(44T^8-422T^7+915T^6+1920T^5-7422T^4+3126T^3+6463T^2-4624T\right)}{6300T^2(T-1)(T-3)^2\left(-T-2T^2\omega_i+2T^2\omega_i^2+2T\omega_i-1\right)^2} \\
& +\frac{\omega_i^2\left(44T^9-686T^8+3503T^7-4640T^6-9523T^5+25\,213T^4-13\,128T^3-1263T^2\right)}{6300T^2(T-1)(T-3)^2\left(-T-2T^2\omega_i+2T^2\omega_i^2+2T\omega_i-1\right)^2} \\
& +\frac{\omega_i^3\left(2684T^8-220T^9-10\,778T^7+10\,928T^6+15\,400T^5-26\,524T^4+8510T^3\right)}{6300T^2(T-1)(T-3)^2\left(-T-2T^2\omega_i+2T^2\omega_i^2+2T\omega_i-1\right)^2} \\
& +\frac{\omega_i^4\left(452T^9-5210T^8+17\,309T^7-13\,115T^6-10\,645T^5+8185T^4\right)}{6300T^2(T-1)(T-3)^2\left(-T-2T^2\omega_i+2T^2\omega_i^2+2T\omega_i-1\right)^2} \\
& +\frac{\omega_i^5\left(5976T^8-444T^9-16\,056T^7+7320T^6+3204T^5\right)}{6300T^2(T-1)(T-3)^2\left(-T-2T^2\omega_i+2T^2\omega_i^2+2T\omega_i-1\right)^2} \\
& +\frac{\omega_i^6\left(204T^9-4476T^8+8604T^7-1452T^6\right)}{6300T^2(T-1)(T-3)^2\left(-T-2T^2\omega_i+2T^2\omega_i^2+2T\omega_i-1\right)^2} \\
& +\frac{\omega_i^7\left(2208T^8-48T^9-2160T^7\right)}{6300T^2(T-1)(T-3)^2\left(-T-2T^2\omega_i+2T^2\omega_i^2+2T\omega_i-1\right)^2} \\
& +\frac{\omega_i^8\left(12T^9-540T^8\right)}{6300T^2(T-1)(T-3)^2\left(-T-2T^2\omega_i+2T^2\omega_i^2+2T\omega_i-1\right)^2};
\end{aligned}
$$

Model 3

$$
\xi_{i,3} = \frac{2T^2\omega_i^2-2T^2\omega_i+T^2-8}{15T\left(T-4\right)},
$$

$$
\begin{aligned}
\varsigma_{i,3}^2 \quad = \quad & \frac{11}{6300}\Big(\frac{2\omega_i^4(T^5-\frac{156}{11}T^4)-4\omega_i^3(T^5-\frac{156}{11}T^4)}{T^2\left(T-2\right)\left(T-4\right)^2} \\
& \frac{+6\omega_i^2(T^5-\frac{356}{33}T^4+\frac{166}{11}T^2+\frac{137}{33}T^3)}{T^2\left(T-2\right)\left(T-4\right)^2} \\
& \frac{-6\omega_i(\frac{2}{3}T^5-\frac{200}{33}T^4+\frac{137}{33}T^3+\frac{166}{11}T^2)}{T^2\left(T-2\right)\left(T-4\right)^2} \\
& \frac{+T^5-\frac{100}{11}T^4+\frac{137}{11}T^3+\frac{348}{11}T^2-\frac{1448}{11}T+\frac{2208}{11}}{T^2\left(T-2\right)\left(T-4\right)^2}\Big).
\end{aligned}
$$

Proof. See the Appendix. ■

Since $\eta_{i,T,k}(\omega_i)$ are *iid* with expectation $\xi_{i,k}$ and variance $\varsigma^2_{i,k}$ for fixed T, it is easily seen by the Lindberg-Levy central limit theorem that for sufficiently large N,

$$Z_k(\omega) = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \frac{(\widehat{\eta_{i,T,k}}(\omega_i) - \xi_{i,k})}{\zeta_{i,k}} \Rightarrow N(0,1). \tag{12}$$

Theorem 2. *Under Assumption 2.1,2.2 and under the null hypothesis (6), for fixed T and as $N \to \infty$, the statistic $Z_k(\omega)$ defined by (12) converges to the standard normal variate.*
Proof. Standard. ■

Remark 1 *The tests can be applied to unbalanced panels. The results derived for balanced panels will essentially remain the same. When individual units have different sample size T_i, the mean and variance now depend on T_i, but when we standardize $\widehat{\eta_{i,T_i,k}}(\omega_i)$ as $(\widehat{\eta_{i,T_i,k}}(\omega_i) - \xi_{i,k,T_i})/\zeta_{i,k,T_i}$ the proofs for Theorem 2 are unchanged.*

Remark 2 *We observe that when $T \to \infty$, the corresponding asymptotic moments in Theorem 1 are consistent with the moments suggested by Hadri and Rao (2007).*

Remark 3 *As in Hadri and Larsson (2005), it is easy to show, using simulations, that the power of the tests increases as the proportion of unit roots increases under the alternative.*

Remark 4 *In the case where ϵ_{it} is serially correlated, we need first to determine the order of the serial correlation, then we should derive the two first moments of our statistic incorporating this information, lastly we must replace $\widehat{\sigma}^2_{\epsilon,i}$ by a consistent estimator of the long-run variance of ϵ_{it}.*

Remark 5 *In the likely case where there is cross-sectional dependency, one readily available method that may be used to correct for it is the bootstrap as shown inter alia by Maddala and Wu (1999), Wu and Wu (2001) and Chang (2004).*

4 Testing for stationarity with an unknown break date

In this Section, we consider the more realistic case where the break date is unknown. Therefore, we have to estimate the break date and construct the test statistic using the estimated break point. Different approaches of estimating the break have been suggested in the literature. In this paper, we use the least-squares method suggested by Bai (1994, 1997) and Kurozumi (2002). This method considers using the estimate of the break date that minimizes the sum of squared residuals (SSR) from the relevant regression under the null hypothesis, that is,

$$\hat{\omega}_i = \arg \min_{0<\omega_i<1} (SSR(\omega_i)).$$

Bai (1994, 1997) show that the estimated fraction of the pre-break point, defined by $\widehat{\omega}_i = T_{B,i}/T_i$, is consistent when ϵ_{it} is an $I(0)$ stochastic process.

Since we allow for different break locations across individuals, we need to detect the break in each one of the individual time series. Therefore, after $\hat{\omega}_i's$, $i = 1, ..., N$ are obtained, we only need to replace ω_i by $\hat{\omega}_i$ and hence obtain

$$Z_k(\widehat{\omega}) = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \frac{(\widehat{\eta_{i,T,k}}(\widehat{\omega}_i) - \xi_{i,k}(\widehat{\omega}_i))}{\zeta_{i,k}(\widehat{\omega}_i)}, \tag{13}$$

where $\xi_{i,k}$ and $\zeta_{i,k}$ follow the same definition as in Theorem 1 and calculated using the consistent estimator $\widehat{\omega}_i$ instead of ω_i. Thus, we perform the hypothesis testing as if the estimated break point were known.

5 Monte Carlo Simulation results.

In this section, we conducted Monte Carlo simulations to investigate the finite sample properties of our proposed statistics. All simulation results are based on 5,000 replications and we use the critical value of 1.645 (5% significance level). It is evident that the distributions of our statistics under the null do not depend on $\alpha_i s$, $\delta_i s$, $\beta_i s$, $\gamma_i s$ and $\sigma^2_{\epsilon,i} s$. The data-generating processes (DGP) under the null hypothesis are given by (1)-(4) with $\alpha_i \sim U[0, 10]$, $\delta_i \sim U[0, 10]$, $\beta_i \sim U[0, 2]$ and $\gamma_i \sim U[0, 5]$, where $U[.]$ denotes the uniform distributions. The sample sizes are given as combinations of different N and T. For each sample size, the parameters of the DGP are generated once and fixed in all the replications. The break fraction is randomly generated as $\omega_i \sim U[0.15, 0.85]$ with a 15% trimming at both ends of the time series. We also investigate the consequences for inference in finite samples of assuming that T is asymptotic rather than fixed. This enables us to evaluate the appropriate size of T that can be reasonably considered to be asymptotic. Size and power results for the cases with a known break date and an unknown break date are reported in the following tables.

Table 1 presents the empirical sizes of $Z_k(\omega)$ for T fixed and for T assumed asymptotic. We note that there is a considerable improvement in the size when the exact finite-sample moments are used, particularly for T small and N large. This makes the tests more suitable for micro panels. The size differences between the two asymptotics decrease as expected when T increases. However, the tests with fixed T register less size distortion even with $T = 100$.

The size of the statistic when the break date is unknown is reported in Table 2. In general, the size is very close to the nominal value although for model 1 and 2, size distortions appear to increase with N when we assume T fixed. The size distortions for models 1 and 2 have also been noted by Kurozumi (2002) in a time series context and by Hadri and Rao (2007) in panel data.

[Table 1 and 2 here]

The power results of the test statistic with a known break date and an estimated break date are shown in Table 3 and 4 respectively. Under the alternative hypothesis, we allow for different proportions of unit root processes ($M = N_1/N$) in the panel. To save space, we only report the simulations under the condition that all the cross sections follow a unit root, that is, $M = 1$. As a function of T and N, the power also changes with different λ ($\lambda = \frac{\sigma_u^2}{\sigma_\epsilon^2}$). We recall that $\lambda = 0$ ($\sigma_u^2 = 0$) means that y is stationary whereas $\lambda = \infty$ entails that y comprises a random walk. By varying the value of λ we can see how the power of the test changes as we approach the two polar cases (stationary versus nonstationary y). We set $\lambda = 0.01$ and 0.03 in our simulations. In general, the power of the tests increases as T or N or both get larger and increases with λ for all T and N. This is also true when the break date is unknown. We found that when the break date is estimated, the power decreases relatively in some cases because of the fact that the estimation of the break induces power losses.

[Table 3 and 4 here]

Table 1. Empirical Size when the break point is known[2]
Model 0

	$N=5$	$N=10$	$N=15$	$N=25$	$N=50$	$N=100$
$T=10$	0.0380	0.0604	0.0560	0.0546	0.0466	0.0556
	(*0.1516*)	(*0.1924*)	(*0.2656*)	(*0.3292*)	(*0.5176*)	(*0.8230*)
$T=15$	0.0738	0.0756	0.0742	0.0706	0.0666	0.0508
	(*0.1254*)	(*0.1488*)	(*0.1880*)	(*0.2364*)	(*0.3540*)	(*0.5142*)
$T=25$	0.0576	0.0608	0.0662	0.0580	0.0558	0.0464
	(*0.0766*)	(*0.0976*)	(*0.1130*)	(*0.1326*)	(*0.1724*)	(*0.2508*)
$T=50$	0.0646	0.0610	0.0644	0.0570	0.0642	0.0540
	(*0.0760*)	(*0.0784*)	(*0.0884*)	(*0.0922*)	(*0.1082*)	(*0.1340*)
$T=100$	0.0714	00598	0.0614	0.0616	0.0598	0.0574
	(*0.0752*)	(*0.0706*)	(*0.076*)	(*0.0730*)	(*0.0766*)	(*0.0916*)

[2]Size based on one sided *N(0,1)*, critical value 1.645 (5%).
Italic indicates the size of the test when asymptotic means and variances are used.

Model 1

	$N=5$	$N=10$	$N=15$	$N=25$	$N=50$	$N=100$
$T=10$	0.0714	0.0836	0.0986	0.0866	0.0804	0.0656
	(*0.2730*)	(*0.4720*)	(*0.6456*)	(*0.8790*)	(*0.9932*)	(*1.0000*)
$T=15$	0.1052	0.0862	0.0640	0.0656	0.0694	0.0724
	(*0.1530*)	(*0.2442*)	(*0.3496*)	(*0.5238*)	(*0.8114*)	(*0.9818*)
$T=25$	0.0824	0.0826	0.0826	0.0836	0.0758	0.0684
	(*0.1096*)	(*0.1502*)	(*0.1996*)	(*0.2700*)	(*0.4124*)	(*0.6650*)
$T=50$	0.0858	0.0888	0.0736	0.0822	0.0754	0.0716
	(*0.0858*)	(*0.1002*)	(*0.1044*)	(*0.1350*)	(*0.1766*)	(*0.2532*)
$T=100$	0.0768	0.0682	0.0610	0.0734	0.0672	0.0696
	(*0.0734*)	(*0.0738*)	(*0.0760*)	(*0.0880*)	(*0.0974*)	(*0.1326*)

Model 2

	$N=5$	$N=10$	$N=15$	$N=25$	$N=50$	$N=100$
$T=10$	0.1320	0.0604	0.0564	0.1326	0.0502	0.0700
	(*0.4046*)	(*0.5864*)	(*0.7848*)	(*0.9652*)	(*0.9994*)	(*1.0000*)
$T=15$	0.0740	0.0642	0.0776	0.0764	0.0638	0.0744
	(*0.1996*)	(*0.2976*)	(*0.4356*)	(*0.6258*)	(*0.8912*)	(*0.9966*)
$T=25$	0.0588	0.0622	0.0648	0.0700	0.0624	0.0794
	(*0.1048*)	(*0.1516*)	(*0.1984*)	(*0.2946*)	(*0.4414*)	(*0.7332*)
$T=50$	0.0684	0.0578	0.0546	0.0672	0.0630	0.0530
	(*0.0854*)	(*0.0972*)	(*0.1034*)	(*0.1408*)	(*0.1826*)	(*0.2784*)
$T=100$	0.0640	0.0620	0.0598	0.0618	0.0504	0.0558
	(*0.0744*)	(*0.0726*)	(*0.0804*)	(*0.0920*)	(*0.0928*)	(*0.1290*)

Model 3

	$N=5$	$N=10$	$N=15$	$N=25$	$N=50$	$N=100$
$T=10$	0.0920	0.0604	0.0718	0.0482	0.0390	0.0502
	(0.5656)	*(0.8486)*	*(0.9848)*	*(0.9986)*	*(1.0000)*	*(1.0000)*
$T=15$	0.0912	0.0746	0.0852	0.0844	0.0742	0.0414
	(0.3296)	*(0.4976)*	*(0.6772)*	*(0.8656)*	*(0.9898)*	*(1.0000)*
$T=25$	0.0534	0.0584	0.0566	0.0620	0.0562	0.0514
	(0.1378)	*(0.2134)*	*(0.2816)*	*(0.4302)*	*(0.6726)*	*(0.9130)*
$T=50$	0.0576	0.0650	0.0608	0.0610	0.0616	0.0560
	(0.0940)	*(0.1184)*	*(0.1438)*	*(0.1804)*	*(0.2718)*	*(0.4330)*
$T=100$	0.0628	0.0528	0.0616	0.0620	0.0534	0.0562
	(0.0790)	*(0.0762)*	*(0.0916)*	*0.1076()*	*(0.1290)*	*(0.1796)*

Table 2. Empirical size when the break point is unknown

Model 0

	$N=5$	$N=10$	$N=15$	$N=25$	$N=50$	$N=100$
$T=10$	0.0648	0.0460	0.0394	0.0344	0.0308	0.0156
	(*0.1230*)	(*0.1498*)	(*0.1870*)	(*0.2718*)	(*0.5026*)	(*0.7386*)
$T=15$	0.0580	0.0394	0.0412	0.0528	0.0288	0.0214
	(*0.0812*)	(*0.0822*)	(*0.1132*)	(*0.1948*)	(*0.2278*)	(*0.3526*)
$T=25$	0.0702	0.0534	0.0470	0.0536	0.0286	0.0306
	(*0.0890*)	(*0.0852*)	(*0.0842*)	(*0.1028*)	(*0.0994*)	(*0.1746*)
$T=50$	0.0656	0.0534	0.0444	0.0514	0.0444	0.0260
	(*0.0720*)	(*0.0668*)	(*0.0578*)	(*0.0814*)	(*0.0806*)	(*0.0592*)
$T=100$	0.0696	0.0570	0.0518	0.0542	0.0414	0.0368
	(*0.0716*)	(*0.0642*)	(*0.0598*)	(*0.0630*)	(*0.0570*)	(*0.0584*)

Model 1

	$N=5$	$N=10$	$N=15$	$N=25$	$N=50$	$N=100$
$T=10$	0.0626	0.0698	0.0274	0.0420	0.0160	0.0092
	(*0.2324*)	(*0.4080*)	(*0.5344*)	(*0.7618*)	(*0.9806*)	(*1.0000*)
$T=15$	0.0590	0.0378	0.0442	0.0270	0.0098	0.0116
	(*0.1210*)	(*0.1550*)	(*0.2650*)	(*0.4014*)	(*0.5226*)	(*0.9120*)
$T=25$	0.0420	0.0340	0.0610	0.0266	0.0214	0.0142
	(*0.0798*)	(*0.0836*)	(*0.1478*)	(*0.1434*)	(*0.2128*)	(*0.3970*)
$T=50$	0.0516	0.0428	0.0406	0.0220	0.0284	0.0118
	(*0.0638*)	(*0.0586*)	(*0.0716*)	(*0.0538*)	(*0.1018*)	(*0.1034*)
$T=100$	0.0496	0.0646	0.0336	0.0252	0.0204	0.0190
	(*0.0548*)	(*0.0758*)	(*0.0466*)	(*0.0456*)	(*0.0438*)	(*0.0554*)

Model 2

	$N=5$	$N=10$	$N=15$	$N=25$	$N=50$	$N=100$
$T=10$	0.0466	0.0074	0.0148	0.0068	0.0008	0.0004
	(*0.2396*)	(*0.2954*)	(*0.5306*)	(*0.7846*)	(*0.9682*)	(*1.0000*)
$T=15$	0.0360	0.0152	0.0262	0.0056	0.0030	0.0006
	(*0.1352*)	(*0.1304*)	(*0.2534*)	(*0.2910*)	(*0.5380*)	(*0.7840*)
$T=25$	0.0434	0.0178	0.0086	0.0194	0.0108	0.0020
	(*0.0674*)	(*0.0608*)	(*0.0446*)	(*0.1240*)	(*0.2042*)	(*0.2722*)
$T=50$	0.0510	0.0158	0.0370	0.0280	0.0076	0.0044
	(*0.0726*)	(*0.0312*)	(*0.0722*)	(*0.0710*)	(*0.0388*)	(*0.0466*)
$T=100$	0.0576	0.0518	0.0342	0.0368	0.0200	0.0184
	(*0.0666*)	(*0.0632*)	(*0.0466*)	(*0.0536*)	(*0.0398*)	(*0.0480*)

Model 3

	$N = 5$	$N = 10$	$N = 15$	$N = 25$	$N = 50$	$N = 100$
$T = 10$	0.0658	0.0560	0.0494	0.0446	0.0366	0.0318
	(0.5016)	*(0.8154)*	*(0.9528)*	*(0.9978)*	*(1.0000)*	*(1.0000)*
$T = 15$	0.0656	0.0490	0.0524	0.0460	0.0308	0.0332
	(0.2716)	*(0.4368)*	*(0.5870)*	*(0.7714)*	*(0.9826)*	*(1.0000)*
$T = 25$	0.0672	0.0538	0.0466	0.0556	0.0462	0.0460
	(0.1564)	*(0.1956)*	*(0.2658)*	*(0.4038)*	*(0.6664)*	*(0.8992)*
$T = 50$	0.0600	0.0658	0.0586	0.0524	0.0494	0.0452
	(0.0932)	*(0.1200)*	*(0.1326)*	*(0.1650)*	*(0.2632)*	*(0.3954)*
$T = 100$	0.0676	0.0630	0.0592	0.0514	0.0566	0.0520
	(0.0824)	*(0.0836)*	*(0.0896)*	*(0.0922)*	*(0.1278)*	*(0.1752)*

Table 3. Power of the statistics with a known break

Model 0							
λ		$N=5$	$N=10$	$N=15$	$N=25$	$N=50$	$N=100$
0.01	$T=10$	0.0734	0.1046	0.0514	0.0704	0.0852	0.0826
	$T=15$	0.1042	0.0604	0.1464	0.1440	0.1502	0.1934
	$T=25$	0.1372	0.1158	0.2184	0.2544	0.4354	0.6232
	$T=50$	0.3916	05060	0.6602	0.9738	1.0000	1.0000
	$T=100$	0.9918	0.9992	1.0000	1.0000	1.0000	1.0000
0.03	$T=10$	0.0856	0.1438	0.0710	0.0986	0.1804	0.2010
	$T=15$	0.1800	0.0754	0.3484	0.3234	0.4068	0.6372
	$T=25$	0.3114	0.2748	0.6114	0.7292	0.9744	0.9992
	$T=50$	0.8722	0.9710	0.9976	1.0000	1.0000	1.0000
	$T=100$	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

Model 1							
λ		$N=5$	$N=10$	$N=15$	$N=25$	$N=50$	$N=100$
0.01	$T=10$	0.0714	0.0668	0.0808	0.0700	0.0886	0.0974
	$T=15$	0.0798	0.1086	0.0814	0.0898	0.1114	0.1232
	$T=25$	0.0900	0.0854	0.1286	0.1574	0.2030	0.2718
	$T=50$	0.3490	0.2528	0.2526	0.4124	0.7838	0.9498
	$T=100$	0.8822	0.8470	0.9996	1.0000	1.0000	1.0000
0.03	$T=10$	0.0718	0.0752	0.1008	0.0860	0.1204	0.1270
	$T=15$	0.0908	0.1348	0.1248	0.1228	0.1966	0.2542
	$T=25$	0.1306	0.1542	0.2786	0.3686	0.5568	0.7964
	$T=50$	0.8002	0.6956	0.6838	0.9468	1.0000	1.0000
	$T=100$	0.9998	1.0000	1.0000	1.0000	1.0000	1.0000

Model 2							
λ		$N=5$	$N=10$	$N=15$	$N=25$	$N=50$	$N=100$
0.01	$T=10$	0.1364	0.0208	0.0382	0.0532	0.0918	0.0476
	$T=15$	0.0630	0.0674	0.0484	0.0854	0.0894	0.0864
	$T=25$	0.0698	0.0828	0.0898	0.1076	0.1414	0.1846
	$T=50$	0.1474	0.2306	0.1652	0.2852	0.7600	0.8244
	$T=100$	0.6886	0.8866	0.9986	0.9950	1.0000	1.0000
0.03	$T=10$	0.1402	0.0252	0.0446	0.0720	0.1156	0.0638
	$T=15$	0.0748	0.0912	0.0740	0.1124	0.1382	0.1632
	$T=25$	0.1174	0.1176	0.1590	0.2552	0.4298	0.5770
	$T=50$	0.3376	0.6526	0.5046	0.8426	0.9998	1.0000
	$T=100$	0.9988	1.0000	1.0000	1.0000	1.0000	1.0000

		Model 3					
λ		$N = 5$	$N = 10$	$N = 15$	$N = 25$	$N = 50$	$N = 100$
0.01	$T = 10$	0.0918	0.1254	0.0304	0.0720	0.0572	0.0392
	$T = 15$	0.0718	0.0572	0.0808	0.1188	0.0804	0.0858
	$T = 25$	0.0880	0.0690	0.1052	0.0984	0.1422	0.1542
	$T = 50$	0.1556	0.1484	0.1466	0.2438	0.6456	0.7338
	$T = 100$	0.6838	0.8040	0.9842	0.9778	0.9998	1.0000
0.03	$T = 10$	0.0938	0.1332	0.0378	0.0854	0.0684	0.0452
	$T = 15$	0.0852	0.0738	0.1106	0.1408	0.1112	0.1448
	$T = 25$	0.1294	0.0884	0.1984	0.1834	0.3816	0.4562
	$T = 50$	0.3972	0.4104	0.4162	0.7660	0.9986	1.0000
	$T = 100$	0.9990	1.0000	1.0000	1.0000	1.0000	1.0000

Table 4. Power of the test with an unknown break

λ		$N=5$	$N=10$	$N=15$	$N=25$	$N=50$	$N=100$
				Model 0			
0.01	$T=10$	0.0640	0.0632	0.0432	0.0510	0.0514	0.0422
	$T=15$	0.0814	0.0648	0.0526	0.0586	0.0940	0.0850
	$T=25$	0.1096	0.1640	0.1242	0.2986	0.2634	0.2806
	$T=50$	0.3748	0.9160	0.6516	0.8366	0.9996	1.0000
	$T=100$	0.9980	0.9076	1.0000	1.0000	1.0000	1.0000
0.03	$T=10$	0.0748	0.0940	0.0642	0.0810	0.0948	0.1074
	$T=15$	0.1558	0.1188	0.1074	0.1288	0.2842	0.3278
	$T=25$	0.2696	0.4178	0.3204	0.8400	0.8246	0.9536
	$T=50$	0.7982	0.9998	0.9976	1.0000	1.0000	1.0000
	$T=100$	1.0000	0.9998	1.0000	1.0000	1.0000	1.0000
				Model 1			
λ		$N=5$	$N=10$	$N=15$	$N=25$	$N=50$	$N=100$
0.01	$T=10$	0.0624	0.0528	0.0502	0.0546	0.0246	0.0234
	$T=15$	0.0824	0.0750	0.0436	0.0742	0.0448	0.0088
	$T=25$	0.0574	0.0558	0.0746	0.0410	0.0536	0.0382
	$T=50$	0.1666	0.3372	0.1632	0.3316	0.6382	0.8198
	$T=100$	0.2746	0.8616	0.9958	0.9996	1.0000	1.0000
0.03	$T=10$	0.0614	0.0554	0.0580	0.0548	0.0300	0.0320
	$T=15$	0.1082	0.0956	0.0526	0.1058	0.0872	0.0182
	$T=25$	0.0782	0.1240	0.1266	0.0840	0.2468	0.2746
	$T=50$	0.4022	0.8852	0.5722	0.9250	0.9998	1.0000
	$T=100$	0.7896	1.0000	1.0000	1.0000	1.0000	1.0000
				Model 2			
λ		$N=5$	$N=10$	$N=15$	$N=25$	$N=50$	$N=100$
0.01	$T=10$	0.0528	0.0232	0.0024	0.0158	0.0158	0.0012
	$T=15$	0.0336	0.0380	0.0120	0.0100	0.0012	0.0002
	$T=25$	0.0392	0.0224	0.0340	0.0364	0.0244	0.0094
	$T=50$	0.1108	0.1206	0.0568	0.2554	0.1642	0.7076
	$T=100$	0.9102	0.6232	0.9404	0.9416	0.9832	1.0000
0.03	$T=10$	0.0584	00250	0.0022	0.0200	0.0012	0.0000
	$T=15$	0.0396	0.0502	0.0140	0.0152	0.0034	0.0006
	$T=25$	0.0598	0.0364	0.0686	0.0896	0.1170	0.0772
	$T=50$	0.3394	0.5062	0.2466	0.8910	0.9008	1.0000
	$T=100$	1.0000	0.9982	1.0000	1.0000	1.0000	1.0000

λ		Model 3					
		$N=5$	$N=10$	$N=15$	$N=25$	$N=50$	$N=100$
0.01	$T=10$	0.0646	0.0512	0.0524	0.0456	0.0366	0.0356
	$T=15$	0.0614	0.0534	0.0524	0.0436	0.0540	0.0464
	$T=25$	0.0808	0.0734	0.0638	0.0838	0.0820	0.1170
	$T=50$	0.1382	0.1764	0.1810	0.3774	0.4536	0.5538
	$T=100$	0.4606	0.4762	0.9300	0.9904	0.9976	1.0000
0.03	$T=10$	0.0664	0.0536	0.0574	0.0480	0.0390	0.0488
	$T=15$	0.0724	0.0638	0.0588	0.0590	0.0826	0.0774
	$T=25$	0.1160	0.1076	0.1084	0.1636	0.2028	0.3718
	$T=50$	0.3264	0.5280	0.5966	0.9222	0.9912	0.9990
	$T=100$	0.9730	0.9798	1.0000	1.0000	1.0000	1.0000

6 Conclusion

In this paper we propose testing for stationarity allowing for a break in heterogeneous panels where the time dimension is finite. The assumption of T finite makes our tests suitable for micro as well as macro panels. Moreover, the test is valid in principle for any (T,N) blend which is not the case when the test is derived under the assumption of $T \longrightarrow \infty$. In the latter case, the validity of the test requires the condition $N/T \longrightarrow 0$, making the test applicable to panels where T is larger than N. The first two finite-sample moments of our tests are derived analytically for all the four models of breaks using two approaches: Laplace Transform and Ghazal's (1994) lemma 1. The limiting distributions of the test statistics are shown to be normal. We also consider the more realistic case where the break location has to be estimated. Monte Carlo simulations show that our proposed tests have generally empirical sizes that are very close to the nominal one, except for models 1 and 2 with unknown break, and their power increases significantly with N, T and λ. The results also show that the assumption that T is fixed rather than asymptotic leads, generally, to tests that are considerably less size distorted.

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7 Appendix

1. Proof of Theorem 1, using Ghazal's (1994) corollary 1

At first, introduce the notation

$$s_n^{(T)} = \sum_{i=1}^{T} i^n, \quad n = 1, 2, ..., \tag{14}$$

where it is known that

$$s_1^{(T)} = \frac{T(T+1)}{2}, \quad (15)$$

$$s_2^{(T)} = \frac{T(T+1)(2T+1)}{6}, \quad (16)$$

$$s_3^{(T)} = \frac{T^2(T+1)^2}{4}, \quad (17)$$

$$s_4^{(T)} = \frac{T(T+1)(2T+1)\left(3T^2+3T-1\right)}{30}. \quad (18)$$

To simplify matters, let us start by assuming ϵ is a white noise process (the index i is omitted in the proof). We want to find the first two moments of

$$S_T = T^{-1}\frac{y'ML'LMy}{y'My},$$

under the hypothesis that $\sigma_u = 0$, i.e. $r_t = r_0 = 0$ for all t. Moreover, since the distribution of S_T does not depend on β, we may without loss of generality set $\beta = 0$, implying $y = \epsilon$. Observe that we gain the special model of Hadri and Larsson (2002) by putting $Z = 1_T \equiv (1, ..., 1)'$.

To find the required moments, we may use the following lemma, which is corollary 1 of Ghazal (1994).

Lemma 6 *Let x be distributed as $N_p(0, I_p)$. Then, if F and G are symmetric $p \times p$ matrices of rank $r < p$ that commute, and if F is idempotent, then*

$$E\left(\frac{x'Gx}{x'Fx}\right) = \frac{\operatorname{tr} G}{r}, \quad (19)$$

$$E\left\{\left(\frac{x'Gx}{x'Fx}\right)^2\right\} = \frac{2\operatorname{tr}(G^2) + (\operatorname{tr} G)^2}{r(r+2)}. \quad (20)$$

It is easy to see that $F = M'M = M$ and $G = M'L'LM$ are symmetric, that they commute and that F is idempotent. As for the rank, the M matrix is idempotent, hence the rank equals

$$\operatorname{tr} M = T - T^{-1}tr(ZZ') = T - p.$$

$$\begin{aligned} \operatorname{tr} G &= \operatorname{tr}(M'L'LM) \\ &= \operatorname{tr}\left(M^2L'L\right) = \operatorname{tr}(ML'L) = \operatorname{tr}\left\{L'L - Z(Z'Z)^{-1}Z'L'L\right\} \\ &= \operatorname{tr}(L'L) - \operatorname{tr}\left\{LZ(Z'Z)^{-1}Z'L'\right\}. \end{aligned}$$

Here, from (14),

$$\operatorname{tr}(L'L) = s_1^{(T)}. \quad (21)$$

Hence, (19) yields

$$E(S_T) = T^{-1}(T-p)^{-1}\left(s_1^{(T)} - s_1^*\right), \quad s_1^* \equiv \operatorname{tr}\left\{LZ(Z'Z)^{-1}Z'L'\right\}. \quad (22)$$

The value of s_1^* depends on the specification of Z.

To find the second moment, since

$$\begin{aligned}
\operatorname{tr}(G^2) &= \operatorname{tr}\{(M'L'LM)^2\} \\
&= \operatorname{tr}\left\{(ML'L)^2\right\} \\
&= \operatorname{tr}\left[\left\{\left(I_T - Z(Z'Z)^{-1}Z'\right)L'L\right\}^2\right] \\
&= \operatorname{tr}\left\{(L'L)^2\right\} - 2\operatorname{tr}\left\{Z(Z'Z)^{-1}Z'\,(L'L)^2\right\} + \operatorname{tr}\left[\left\{LZ(Z'Z)^{-1}Z'L'\right\}^2\right].
\end{aligned}$$

Hence, because

$$\begin{aligned}
\operatorname{tr}\left\{(L'L)^2\right\} &= 1^2\,(2T-1) + 2^2\,(2T-3) + ... + T^2 \cdot 1 \\
&= \frac{1}{6}T\,(T+1)\left(T^2+T+1\right) \equiv b, \qquad (23)
\end{aligned}$$

(cf Hadri and Larsson (2002)), we have

$$\operatorname{tr}\left\{(ML'L)^2\right\} = b - s_2^*,$$

where

$$s_2^* \equiv 2\operatorname{tr}\left\{Z(Z'Z)^{-1}Z'\,(L'L)^2\right\} - \operatorname{tr}\left[\left\{LZ(Z'Z)^{-1}Z'L'\right\}^2\right]. \qquad (24)$$

Thus, according to the (20), we obtain,

$$E\left(S_T^2\right) = T^{-2}\,(T-p+2)^{-1}\,(T-p)^{-1}\left\{2\,(b-s_2^*) + \left(s_1^{(T)} - s_1^*\right)^2\right\}. \qquad (25)$$

The values of s_1^* and s_2^* depend on the choice of Z.

7.0.1 Model 0

As for Model 0, we have $Z = (1_T, d_1)$, where $d_1 \equiv \left(0'_{T_B}, 1'_{T-T_B}\right)'$. Consequently,

$$Z'Z = \begin{pmatrix} 1'_T 1_T & 1'_T d_1 \\ d'_1 1_T & d'_1 d_1 \end{pmatrix} = \begin{pmatrix} T & T-T_B \\ T-T_B & T-T_B \end{pmatrix} = T\begin{pmatrix} 1 & \omega_i \\ \omega_i & \omega_i \end{pmatrix},$$

where $\omega_i \equiv (T-T_B)/T$. Hence, it follows that

$$(Z'Z)^{-1} = T^{-1}\omega_i^{-1}\,(1-\omega_i)^{-1}\begin{pmatrix} \omega_i & -\omega_i \\ -\omega_i & 1 \end{pmatrix}, \qquad (26)$$

and so, introducing $2_c\,(\omega_i B') = \omega_i B' + B\omega'_i$ for arbitrary matrices ω_i and B,

$$\begin{aligned}
Z(Z'Z)^{-1}Z' &= T^{-1}\omega_i^{-1}\,(1-\omega_i)^{-1}\,(1_T, d_1)\begin{pmatrix} \omega_i & -\omega_i \\ -\omega_i & 1 \end{pmatrix}\begin{pmatrix} 1'_T \\ d'_1 \end{pmatrix} \\
&= T^{-1}\omega_i^{-1}\,(1-\omega_i)^{-1}\left\{\omega_i 1_T 1'_T - \omega_i 2_c\,(1_T d'_1) + d_1 d'_1\right\}. \qquad (27)
\end{aligned}$$

Hence, putting

$$L1_T = (1, 2, ..., T)' \equiv \tau, \quad (28)$$
$$Ld_1 \equiv (0, ..., 0, 1, ..., T - T_B)' \equiv \tau_{\omega_i}, \quad (29)$$

we get

$$LZ(Z'Z)^{-1}Z'L' = T^{-1}\omega_i^{-1}(1-\omega_i)^{-1}\left\{\omega_i\tau\tau' - \omega_i 2_c\left(\tau\tau'_{\omega_i}\right) + \tau_{\omega_i}\tau'_{\omega_i}\right\}, \quad (30)$$

implying

$$s_1^* = T^{-1}\omega_i^{-1}(1-\omega_i)^{-1}\left\{\omega_i\left(\tau'\tau - 2\tau'\tau_{\omega_i}\right) + \tau'_{\omega_i}\tau_{\omega_i}\right\}.$$

Here,

$$\tau'\tau = \sum_{i=1}^{T} i^2 = s_2^{(T)}, \quad (31)$$
$$\tau'\tau_{\omega_i} = \sum_{i=1}^{T-T_B}(i+T_B)\,i = s_2^{(\omega_i T)} + (1-\omega_i)\,T s_1^{(\omega_i T)}, \quad (32)$$
$$\tau'_{\omega_i}\tau_{\omega_i} = \sum_{i=1}^{T-T_B} i^2 = s_2^{(\omega_i T)}, \quad (33)$$

and so, using (15), (16) and simplifying, we find

$$s_1^{(T)} - s_1^* = \frac{1}{12}\left\{T^2 - 4 + 4T^2\left(\omega_i - \frac{1}{2}\right)^2\right\}. \quad (34)$$

Hence, inserting into (22) with $p = 2$,

$$E(S_T) = \frac{T+2}{12T} + \frac{T}{3(T-2)}\left(\omega_i - \frac{1}{2}\right)^2.$$

Observe the symmetry in $\omega_i - 1/2$. In particular, if $\omega_i = 0$ or 1, we get

$$E(S_T) = \frac{1}{6}\frac{T^2-2}{T(T-2)},$$

cf the corresponding result of Hadri and Larsson (2002), which is $(T+1)/(6T)$.

To find the second moment we, in view of (25), need to calculate s_2^*. To this end, using

$$\begin{aligned}
L'\tau &= \left(\sum_{i=1}^{T} i, \sum_{i=2}^{T} i, ..., T\right)' \\
&= \frac{1}{2}\left(T\left(T+1\right), \left(T-1\right)\left(T+2\right), ..., 2T\right)' \equiv \upsilon, \qquad (35) \\
L'\tau_{\omega_i} &\equiv \left(\sum_{i=1}^{T-T_B} i, ..., \sum_{i=1}^{T-T_B} i, \sum_{i=2}^{T-T_B} i, ..., T-T_B\right)' \\
&= \frac{1}{2}\left(\omega_i T\left(\omega_i T+1\right), ..., \omega_i T\left(\omega_i T+1\right), \left(\omega_i T-1\right)\left(\omega_i T+2\right), ..., 2\omega_i T\right)' \\
&\equiv \upsilon_{\omega_i}, \qquad (36)
\end{aligned}$$

we find via (30) that

$$\begin{aligned}
L'LZ(Z'Z)^{-1}Z'L'L &= T^{-1}\omega_i^{-1}\left(1-\omega_i\right)^{-1} L'\left\{\omega_i\tau\tau' - \omega_i 2_c\left(\tau\tau'_{\omega_i}\right) + \tau_{\omega_i}\tau'_{\omega_i}\right\} L \\
&= T^{-1}\omega_i^{-1}\left(1-\omega_i\right)^{-1}\left\{\omega_i\upsilon\upsilon' - \omega_i 2_c\left(\upsilon\upsilon'_{\omega_i}\right) + \upsilon_{\omega_i}\upsilon'_{\omega_i}\right\}.
\end{aligned}$$

Hence,

$$\begin{aligned}
\operatorname{tr}\left\{Z(Z'Z)^{-1}Z'\left(L'L\right)^2\right\} &= \operatorname{tr}\left\{L'LZ(Z'Z)^{-1}Z'L'L\right\} \\
&= T^{-1}\omega_i^{-1}\left(1-\omega_i\right)^{-1} c_1, \qquad (37)
\end{aligned}$$

where

$$c_1 \equiv \omega_i\left(\upsilon'\upsilon - 2\upsilon'\upsilon_{\omega_i}\right) + \upsilon'_{\omega_i}\upsilon_{\omega_i},$$

with

$$\begin{aligned}
\upsilon'\upsilon &= \frac{1}{4}\sum_{i=1}^{T}\left(T-i+1\right)^2\left(T+i\right)^2 \\
&= \frac{1}{30}T\left(T+1\right)\left(2T+1\right)\left(2T^2+2T+1\right).
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\upsilon'\upsilon_{\omega_i} \\
= &\frac{1}{4}\omega_i T\left(\omega_i T+1\right)\sum_{i=1}^{(1-\omega_i)T}\left(T-i+1\right)\left(T+i\right) \\
&+\frac{1}{4}\sum_{i=1}^{\omega_i T}\left(\omega_i T-i+1\right)^2\left(\omega_i T+i\right)\left(T+i+\left(1-\omega_i\right)T\right)
\end{aligned}$$

and

$$\upsilon'_{\omega_i}\upsilon_{\omega_i} = \frac{1}{4}\left(1-\omega_i\right)T\left(\omega_i T\right)^2\left(\omega_i T+1\right)^2 + \frac{1}{4}\sum_{i=1}^{\omega_i T}\left(\omega_i T-i+1\right)^2\left(\omega_i T+i\right)^2.$$

Moreover, via (30),

$$\begin{aligned}
& \left\{LZ(Z'Z)^{-1}Z'L'\right\}^2 \\
= \;& T^{-2}\omega_i^{-2}\left(1-\omega_i\right)^{-2}\left\{\omega_i\tau\tau' - \omega_i 2_c\left(\tau\tau'_{\omega_i}\right) + \tau_{\omega_i}\tau'_{\omega_i}\right\}^2 \\
= \;& T^{-2}\omega_i^{-2}\left(1-\omega_i\right)^{-2}\left[\omega_i^2\left\{\left(\tau\tau'\right)^2 - 2_c\left(\left(\tau\tau'\right)2_c\left(\tau\tau'_{\omega_i}\right)\right) + \left(2_c\left(\tau\tau'_{\omega_i}\right)\right)^2\right\}\right. \\
& \left. +\omega_i\left\{2_c\left(\left(\tau\tau'\right)\left(\tau_{\omega_i}\tau'_{\omega_i}\right)\right) - 2_c\left(2_c\left(\tau\tau'_{\omega_i}\right)\left(\tau_{\omega_i}\tau'_{\omega_i}\right)\right)\right\} + \left(\tau_{\omega_i}\tau'_{\omega_i}\right)^2\right],
\end{aligned}$$

implying

$$\mathrm{tr}\left[\left\{LZ(Z'Z)^{-1}Z'L'\right\}^2\right] = T^{-2}\omega_i^{-2}\left(1-\omega_i\right)^{-2}c_2,$$

where

$$\begin{aligned}
c_2 \;\equiv\; & \omega_i^2\left\{\left(\tau'\tau\right)^2 - 4\left(\tau'_{\omega_i}\tau\right)\left(\tau'\tau\right) + 2\left(\tau'_{\omega_i}\tau\right)^2 + 2\left(\tau'_{\omega_i}\tau_{\omega_i}\right)\left(\tau'\tau\right)\right\} \\
& +\omega_i\left\{2\left(\tau'_{\omega_i}\tau\right)^2 - 4\left(\tau'_{\omega_i}\tau\right)\left(\tau'_{\omega_i}\tau_{\omega_i}\right)\right\} + \left(\tau'_{\omega_i}\tau_{\omega_i}\right)^2.
\end{aligned}$$

Here, from (31)-(33), (15) and (16),

$$\begin{aligned}
\tau'\tau &= \frac{1}{6}T\left(T+1\right)\left(2T+1\right), && (38) \\
\tau'_{\omega_i}\tau &= \frac{1}{6}\omega_i T\left(\omega_i T+1\right)\left(2\omega_i T+1\right) + \frac{1}{2}\left(1-\omega_i\right)\omega_i T^2\left(\omega_i T+1\right) \\
&= \frac{1}{6}\omega_i T\left(\omega_i T+1\right)\left\{\left(3-\omega_i\right)T+1\right\}, && (39) \\
\tau'_{\omega_i}\tau_{\omega_i} &= \frac{1}{6}\omega_i T\left(\omega_i T+1\right)\left(2\omega_i T+1\right). && (40)
\end{aligned}$$

Now, (23), (24) and simplifications yield

$$b - s_2^* = \frac{1}{720}\left\{-56 + 10T^2 + T^4 + T^2\left(40+24T^2\right)\left(\omega_i - \frac{1}{2}\right)^2 + 16T^4\left(\omega_i - \frac{1}{2}\right)^4\right\},$$

which via (34) and (25) implies

$$\begin{aligned}
E\left(S_T^2\right) \;=\; & \frac{1}{720}T^{-3}\left(T-2\right)^{-1}\left\{-32 - 20T^2 + 7T^4\right. \\
& \left. +8T^2\left(-10+11T^2\right)\left(\omega_i - \frac{1}{2}\right)^2 + 112T^4\left(\omega_i - \frac{1}{2}\right)^4\right\}.
\end{aligned}$$

7.0.2 Model 1

In Model 1, $Z = (1_T, d_1, \tau/T)$, implying, because

$$
\begin{aligned}
1_T'\tau/T &= \frac{T+1}{2}, \\
d_1'\tau/T &= T^{-1}\sum_{i=1}^{T-T_B}(T_B+i) = \frac{(T-T_B)(T+T_B+1)}{2T} = \frac{1}{2}\omega_i\left\{(2-\omega_i)T+1\right\}, \\
\tau'\tau/T^2 &= \frac{(T+1)(2T+1)}{6T},
\end{aligned}
$$

that

$$
\begin{aligned}
Z'Z &= \begin{pmatrix} 1_T'1_T & 1_T'd_1 & 1_T'\tau/T \\ d_1'1_T & d_1'd_1 & d_1'\tau/T \\ \tau'1_T/T & \tau'd_1/T & \tau'\tau/T^2 \end{pmatrix} \\
&= T\begin{pmatrix} 1 & \omega_i & \frac{1}{2}\left(1+T^{-1}\right) \\ \omega_i & \omega_i & \frac{1}{2}\omega_i\left(2-\omega_i+T^{-1}\right) \\ \frac{1}{2}\left(1+T^{-1}\right) & \frac{1}{2}\omega_i\left(2-\omega_i+T^{-1}\right) & \frac{1}{6}\left(1+T^{-1}\right)\left(2+T^{-1}\right) \end{pmatrix},
\end{aligned}
$$

which has inverse

$$
(Z'Z)^{-1} = \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{12} & c_{22} & c_{23} \\ c_{13} & c_{23} & c_{33} \end{pmatrix},
$$

where the c_{ij} may be expressed in explicit formulae, not given here to save space. Moreover,

$$
\begin{aligned}
Z(Z'Z)^{-1}Z' &= (1_T, d_1, \tau/T)\begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{12} & c_{22} & c_{23} \\ c_{13} & c_{23} & c_{33} \end{pmatrix}\begin{pmatrix} 1_T' \\ d_1' \\ \tau'/T \end{pmatrix} \\
&= c_{11}1_T1_T' + c_{12}2_c(1_Td_1') + T^{-1}c_{13}2_c(1_T\tau') \\
&\quad + c_{22}d_1d_1' + T^{-1}c_{23}2_c(d_1\tau') + T^{-2}c_{33}\tau\tau',
\end{aligned}
$$

and via (28), (29) and

$$
L\tau = \left(1, 3, ..., \frac{1}{2}T(T+1)\right)' \equiv \xi, \tag{41}
$$

we find

$$
\begin{aligned}
LZ(Z'Z)^{-1}Z'L' &= c_{11}\tau\tau' + c_{12}2_c\left(\tau\tau_{\omega_i}'\right) + T^{-1}c_{13}2_c\left(\tau\xi'\right) \\
&\quad + c_{22}\tau_{\omega_i}\tau_{\omega_i}' + T^{-1}c_{23}2_c\left(\tau_{\omega_i}\xi'\right) + T^{-2}c_{33}\xi\xi'. \qquad (42)
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
s_1^* &= c_{11}\tau'\tau + 2c_{12}\tau_{\omega_i}'\tau + 2T^{-1}c_{13}\xi'\tau \\
&\quad + c_{22}\tau_{\omega_i}'\tau_{\omega_i} + 2T^{-1}c_{23}\xi'\tau_{\omega_i} + T^{-2}c_{33}\xi'\xi,
\end{aligned}
$$

where (15), (38)-(40) and

$$\xi'\tau = \frac{1}{2}\sum_{i=1}^{T} i^2 (i+1) = \frac{1}{24}T(3T+1)(T+2)(T+1), \quad (43)$$

$$\begin{aligned}
\xi'\tau_{\omega_i} &= \frac{1}{2}\sum_{i=1}^{T-T_B} (T_B+i)(T_B+i+1)\,i \\
&= \frac{1}{2}\sum_{i=1}^{\omega_i T} ((1-\omega_i)T+i)((1-\omega_i)T+i+1)\,i \\
&= \frac{1}{24}\omega_i T(\omega_i T+1)\left(T^2\omega_i^2-4T^2\omega_i+6T^2-3\omega_i T+10T+2\right), \quad (44)
\end{aligned}$$

$$\xi'\xi = \frac{1}{4}\sum_{i=1}^{T} i^2 (i+1)^2 = \frac{1}{60}T(T+2)(T+1)\left(3T^2+6T+1\right), \quad (45)$$

yield, after simplifications, writing $x \equiv \omega_i - 1/2$,

$$s_1^{(T)} - s_1^* = \frac{1}{120}\left(T^2-4+12T^2x^2\right)^{-1}\left(208-80T^2+7T^4-640T^2x^2+40T^4x^2+240T^4x^4\right).$$

Hence, via (22) with $p=3$,

$$\begin{aligned}
E(S_T) &= \frac{1}{120}T^{-1}(T-3)^{-1}\left(T^2-4+12T^2x^2\right)^{-1} \\
&\quad \left(208-80T^2+7T^4-640T^2x^2+40T^4x^2+240T^4x^4\right).
\end{aligned}$$

As for the second moment, we get from (35), (36) and

$$\begin{aligned}
L'\xi &= \left\{\frac{1}{2}\sum_{i=1}^{T} i(i+1), \frac{1}{2}\sum_{i=2}^{T} i(i+1), ..., \right. \\
&\quad \left. \frac{1}{2}\sum_{i=j}^{T} i(i+1), ..., \frac{1}{2}T(T+1)\right\}' \\
&= \left\{\frac{1}{6}T(T+2)(T+1), \frac{1}{6}(T-1)\left(T^2+4T+6\right), ..., \right. \\
&\quad \left. \frac{1}{6}(T+1-j)\left(T^2+2T+jT+j+j^2\right), ..., \frac{1}{2}T(T+1)\right\}' \\
&\equiv \phi, \quad (46)
\end{aligned}$$

that, via (42),

$$\begin{aligned}
L'LZ(Z'Z)^{-1}Z'L'L &= c_{11}\upsilon\upsilon' + c_{12}2_c\left(\upsilon\upsilon'_{\omega_i}\right) + T^{-1}c_{13}2_c\left(\upsilon\phi'\right) \\
&\quad + c_{22}\upsilon_{\omega_i}\upsilon'_{\omega_i} + T^{-1}c_{23}2_c\left(\upsilon_{\omega_i}\phi'\right) + T^{-2}c_{33}\phi\phi',
\end{aligned}$$

implying

$$
\begin{aligned}
c_1^{(1)} &\equiv \operatorname{tr}\left\{Z\left(Z'Z\right)^{-1}Z'\left(L'L\right)^2\right\} \\
&= c_{11}\upsilon'\upsilon + 2c_{12}\upsilon_{\omega_i}'\upsilon + 2T^{-1}c_{13}\phi'\upsilon \\
&\quad + c_{22}\upsilon_{\omega_i}'\upsilon_{\omega_i} + 2T^{-1}c_{23}\phi'\upsilon_{\omega_i} + T^{-2}c_{33}\phi'\phi.
\end{aligned}
$$

Here, via (35), (36) and (46),

$$
\begin{aligned}
\phi'\upsilon &= \frac{1}{12}\sum_{j=1}^{T}\left(T+1-j\right)^2\left(T^2+2T+jT+j+j^2\right)\left(T+j\right), \\
\phi'\upsilon_{\omega_i} &= \frac{1}{12}\omega_i T\left(\omega_i T+1\right)\sum_{j=1}^{(1-\omega_i)T}\left(T+1-j\right)\left(T^2+2T+jT+j+j^2\right) \\
&\quad + \frac{1}{12}\sum_{j=1}^{\omega_i T}\left(3T^2+3T+3Tj-3T^2\omega_i+j-T\omega_i+j^2-2jT\omega_i+T^2\omega_i^2\right) \\
&\quad \left(\omega_i T-j+1\right)^2\left(\omega_i T+j\right), \\
\phi'\phi &= \frac{1}{36}\sum_{j=1}^{T}\left(T+1-j\right)^2\left(T^2+2T+jT+j+j^2\right)^2.
\end{aligned}
$$

Moreover, from (42),

$$
\begin{aligned}
\left\{LZ\left(Z'Z\right)^{-1}Z'L'\right\}^2 &= \left\{c_{11}\tau\tau' + c_{12}2_c\left(\tau\tau_{\omega_i}'\right) + T^{-1}c_{13}2_c\left(\tau\xi'\right)\right. \\
&\quad \left. + c_{22}\tau_{\omega_i}\tau_{\omega_i}' + T^{-1}c_{23}2_c\left(\tau_{\omega_i}\xi'\right) + T^{-2}c_{33}\xi\xi'\right\}^2,
\end{aligned}
$$

implying

$$
\begin{aligned}
& c_2^{(1)} \\
\equiv\;& \operatorname{tr}\left[\left\{LZ\left(Z'Z\right)^{-1}Z'L'\right\}^2\right] \\
=\;& c_{11}^2\left(\tau'\tau\right)^2 + 4c_{11}c_{12}\left(\tau_{\omega_i}'\tau\right)\left(\tau'\tau\right) + 4T^{-1}c_{11}c_{13}\left(\xi'\tau\right)\left(\tau'\tau\right) + 2c_{11}c_{22}\left(\tau_{\omega_i}'\tau\right)^2 \\
& + 4T^{-1}c_{11}c_{23}\left(\tau_{\omega_i}'\tau\right)\left(\xi'\tau\right) + 2T^{-2}c_{11}c_{33}\left(\tau'\xi\right)^2 + 2c_{12}^2\left\{\left(\tau_{\omega_i}'\tau\right)^2 + \left(\tau_{\omega_i}'\tau_{\omega_i}\right)\left(\tau'\tau\right)\right\} \\
& + 4T^{-1}c_{12}c_{13}\left\{\left(\tau_{\omega_i}'\tau\right)\left(\tau'\xi\right) + \left(\tau'\tau\right)\left(\tau_{\omega_i}'\xi\right)\right\} + 4c_{12}c_{22}\left(\tau_{\omega_i}'\tau\right)\left(\tau_{\omega_i}'\tau_{\omega_i}\right) \\
& + 4T^{-1}c_{12}c_{23}\left\{\left(\tau_{\omega_i}'\tau_{\omega_i}\right)\left(\tau'\xi\right) + \left(\tau_{\omega_i}'\tau\right)\left(\tau_{\omega_i}'\xi\right)\right\} + 4T^{-2}c_{12}c_{33}\left(\tau_{\omega_i}'\xi\right)\left(\tau'\xi\right) \\
& + 2T^{-2}c_{13}^2\left\{\left(\tau'\xi\right)^2 + \left(\tau'\tau\right)\left(\xi'\xi\right)\right\} + 4T^{-1}c_{13}c_{22}\left(\tau_{\omega_i}'\tau\right)\left(\tau_{\omega_i}'\xi\right) \\
& + 4T^{-2}c_{13}c_{23}\left\{\left(\tau_{\omega_i}'\tau\right)\left(\xi'\xi\right) + \left(\tau_{\omega_i}'\xi\right)\left(\tau'\xi\right)\right\} + 4T^{-3}c_{13}c_{33}\left(\tau'\xi\right)\left(\xi'\xi\right) \\
& + c_{22}^2\left(\tau_{\omega_i}'\tau_{\omega_i}\right)^2 + 4T^{-1}c_{22}c_{23}\left(\tau_{\omega_i}'\tau_{\omega_i}\right)\left(\tau_{\omega_i}'\xi\right) + 2T^{-2}c_{22}c_{33}\left(\tau_{\omega_i}'\xi\right)^2 \\
& + 2T^{-2}c_{23}^2\left\{\left(\tau_{\omega_i}'\xi\right)^2 + \left(\tau_{\omega_i}'\tau_{\omega_i}\right)\left(\xi'\xi\right)\right\} + 4T^{-3}c_{23}c_{33}\left(\tau_{\omega_i}'\xi\right)\left(\xi'\xi\right) \\
& + T^{-4}c_{33}^2\left(\xi'\xi\right)^2. \qquad (47)
\end{aligned}
$$

Hence, because by (24),

$$s_2^* = 2c_1^{(1)} - c_2^{(1)},$$

we get via (23), (69) and simplifications that

$$\begin{aligned}
& E\left(S_T^2\right) \\
= \quad & T^{-2}\left(T-1\right)^{-1}\left(T-3\right)^{-1}\left\{\left(s_1^{(T)} - s_1^*\right)^2 + 2\left(b - s_2^*\right)\right\} \\
= \quad & \left[50400T^2(T-1)\left(T-3\right)\left\{T^2\left(1+12x^2\right)-4\right\}^2\right]^{-1} \\
& \left\{-3968 + T^2 f_2^{(1)}\left(x\right) + T^4 f_4^{(1)}\left(x\right) + T^6 f_6^{(1)}\left(x\right) + T^8 f_8^{(1)}\left(x\right)\right\},
\end{aligned}$$

with $x = \omega_i - 1/2$,

$$\begin{aligned}
f_2^{(1)}\left(x\right) &\equiv 160\left(-145+52x^2\right), \\
f_4^{(1)}\left(x\right) &\equiv 56\left(291+1960x^2+4400x^4\right), \\
f_6^{(1)}\left(x\right) &\equiv -10\left(355+3388x^2+26768x^4+65856x^6\right), \\
f_8^{(1)}\left(x\right) &\equiv 247+1480x^2+31360x^4+94080x^6+241920x^8.
\end{aligned}$$

7.0.3 Model 2

As for Model 2, we have $Z = (1_T, \tau_{\omega_i}/T, \tau/T)$. Moreover, denoting the entries of $\left(Z'Z\right)^{-1}$ by $c_{ij}^{(2)}$ for $i, j = 1, 2, 3$, we get as in Model 1

$$\begin{aligned}
Z\left(Z'Z\right)^{-1}Z' \quad = \quad & c_{11}^{(2)} 1_T 1_T' + T^{-1} c_{12}^{(2)} 2_c\left(1_T \tau_{\omega_i}'\right) + T^{-1} c_{13}^{(2)} 2_c\left(1_T \tau'\right) \\
& + T^{-2} c_{22}^{(2)} \tau_{\omega_i} \tau_{\omega_i}' + T^{-2} c_{23}^{(2)} 2_c\left(\tau \tau_{\omega_i}'\right) + T^{-2} c_{33}^{(2)} \tau\tau'.
\end{aligned}$$

Moreover, (28), (41) and

$$L\tau_{\omega_i} = \left(0, ..., 0, 1, 3, ..., \frac{1}{2}\omega_i T\left(\omega_i T + 1\right)\right)' \equiv \xi_{\omega_i} \tag{48}$$

yield

$$\begin{aligned}
LZ\left(Z'Z\right)^{-1}Z'L' \quad = \quad & c_{11}^{(2)} \tau\tau' + T^{-1} c_{12}^{(2)} 2_c\left(\tau \xi_{\omega_i}'\right) + T^{-1} c_{13}^{(2)} 2_c\left(\tau\xi'\right) \\
& + T^{-2} c_{22}^{(2)} \xi_{\omega_i} \xi_{\omega_i}' + T^{-2} c_{23}^{(2)} 2_c\left(\xi \xi_{\omega_i}'\right) + T^{-2} c_{33}^{(2)} \xi\xi', \qquad (49)
\end{aligned}$$

implying

$$\begin{aligned}
s_1^* \quad = \quad & c_{11}^{(2)} \tau'\tau + 2T^{-1} c_{12}^{(2)} \xi_{\omega_i}' \tau + 2T^{-1} c_{13}^{(2)} \xi'\tau \\
& + T^{-2} c_{22}^{(2)} \xi_{\omega_i}' \xi_{\omega_i} + 2T^{-2} c_{23}^{(2)} \xi_{\omega_i}' \xi + T^{-2} c_{33}^{(2)} \xi'\xi.
\end{aligned}$$

Here,

$$
\begin{aligned}
\xi_{\omega_i}'\tau &= \frac{1}{2}\sum_{i=1}^{T-T_B} i\,(i+1)\,(T_B+i) \\
&= \frac{1}{24}\,(T-T_B)\,(T-T_B+1)\,(T-T_B+2)\,(T_B+1+3T) \\
&= \frac{1}{24}\omega_i T\,(\omega_i T+1)\,(\omega_i T+2)\,\{(4-\omega_i)\,T+1\}\,, \\
\xi_{\omega_i}'\xi &= \frac{1}{4}\sum_{i=1}^{T-T_B} i\,(i+1)\,(T_B+i)\,(T_B+i+1) \\
&= \frac{1}{120}\,(T-T_B)\,(T-T_B+1)\,(T-T_B+2)\left(T_B^2+3T_BT+3T_B+2+6T^2+12T\right) \\
&= \frac{1}{120}\omega_i T\,(\omega_i T+1)\,(\omega_i T+2)\left(10T^2-5\omega_i T^2+\omega_i^2T^2+15T-3\omega_i T+2\right), \\
\xi_{\omega_i}'\xi_{\omega_i} &= \frac{1}{4}\sum_{i=1}^{\omega_i T} i^2\,(i+1)^2 = \frac{1}{60}\omega_i T\,(\omega_i T+1)\,(\omega_i T+2)\left(3\omega_i^2T^2+1+6\omega_i T\right),
\end{aligned}
$$

which together with (38), (43) and (45) yields, inserted into (22),

$$
\begin{aligned}
& E\,(S_T) \\
=\; & T^{-1}\,(T-3)^{-1}\left(s_1^{(T)}-s_1^*\right) \\
=\; & \left\{120T\,(T-3)\left(-2-T^2+4Tx+4T^2x^2\right)\right\}^{-1} \\
& \left(112+28T^2-5T^4-224Tx+8T^3x-176T^2x^2+8T^4x^2+96T^3x^3+48T^4x^4\right).
\end{aligned}
$$

(Observe that the symmetry in x is no longer present.) As for the second moment, we get via (35), (46) and

$$
\begin{aligned}
L'\xi_{\omega_i} &= \frac{1}{2}\left\{\sum_{i=1}^{\omega_i T} i\,(i+1)\,,...,\sum_{i=1}^{\omega_i T} i\,(i+1)\,,\right. \\
&\qquad \left.\sum_{i=2}^{\omega_i T} i\,(i+1)\,,...,\sum_{i=j}^{\omega_i T} i\,(i+1)\,,...,\omega_i T\,(\omega_i T+1)\right\}' \\
&= \left\{\frac{1}{6}\omega_i T\,(\omega_i T+2)\,(\omega_i T+1)\,,...,\frac{1}{6}\omega_i T\,(\omega_i T+2)\,(\omega_i T+1)\,,\right. \\
&\qquad \frac{1}{6}\,(\omega_i T-1)\left(\omega_i^2T^2+4\omega_i T+6\right),..., \\
&\qquad \left.\frac{1}{6}\,(\omega_i T+1-j)\left(\omega_i^2T^2+(2+j)\,\omega_i T+j+j^2\right),...,\omega_i T\,(\omega_i T+1)\right\}' \\
&\equiv \phi_{\omega_i} \qquad (50)
\end{aligned}
$$

that, from (49),

$$
\begin{aligned}
L'LZ\left(Z'Z\right)^{-1}Z'L'L &= c_{11}\upsilon\upsilon' + T^{-1}c_{12}2_c\left(\upsilon\phi'_{\omega_i}\right) + T^{-1}c_{13}2_c\left(\upsilon\phi'\right) \\
&\quad + T^{-2}c_{22}\phi_{\omega_i}\phi'_{\omega_i} + T^{-2}c_{23}2_c\left(\phi\phi'_{\omega_i}\right) + T^{-2}c_{33}\phi\phi',
\end{aligned}
$$

which yields

$$
\begin{aligned}
c_1^{(2)} &\equiv \operatorname{tr}\left\{Z\left(Z'Z\right)^{-1}Z'\left(L'L\right)^2\right\} \\
&= c_{11}\upsilon'\upsilon + 2T^{-1}c_{12}\phi'_{\omega_i}\upsilon + 2T^{-1}c_{13}\phi'\upsilon \\
&\quad + T^{-2}c_{22}\phi'_{\omega_i}\phi_{\omega_i} + 2T^{-2}c_{23}\phi'_{\omega_i}\phi + T^{-2}c_{33}\phi'\phi,
\end{aligned}
$$

where, using (35), (46) and (50),

$$
\begin{aligned}
\phi'_{\omega_i}\upsilon &\equiv \frac{1}{12}\omega_i T\left(\omega_i T+2\right)\left(\omega_i T+1\right)\sum_{i=1}^{(1-\omega_i)T}\left(T-i+1\right)\left(T+i\right) \\
&\quad + \frac{1}{12}\sum_{i=1}^{\omega_i T}\left(\omega_i T+1-i\right)^2\left\{\omega_i^2T^2+\left(2+i\right)\omega_i T+i+i^2\right\}\left\{\left(2-\omega_i\right)T+i\right\}, \\
\phi'_{\omega_i}\phi &\equiv \frac{1}{36}\omega_i T\left(\omega_i T+2\right)\left(\omega_i T+1\right)\sum_{i=1}^{(1-\omega_i)T}\left(T+1-i\right)\left(T^2+2T+iT+i+i^2\right) \\
&\quad + \frac{1}{36}\sum_{i=1}^{\omega_i T}\left(\omega_i T+1-i\right)^2\left\{\omega_i^2T^2+\left(2+i\right)\omega_i T+i+i^2\right\} \\
&\quad \left(3T^2+3T+3Ti-3T^2\omega_i+i-\omega_i T+i^2-2\omega_i iT+\omega_i^2T^2\right), \\
\phi'_{\omega_i}\phi_{\omega_i} &\equiv \frac{1}{36}\omega_i^2T^2\left(\omega_i T+2\right)^2\left(\omega_i T+1\right)^2\left(1-\omega_i\right)T \\
&\quad + \frac{1}{36}\sum_{i=1}^{\omega_i T}\left(\omega_i T+1-i\right)^2\left\{\omega_i^2T^2+\left(2+i\right)\omega_i T+i+i^2\right\}^2.
\end{aligned}
$$

Further, via (49) we get the corresponding expression to (47) for the corresponding quantity $c_2^{(2)}$, inserting $T^{-1}\xi_{\omega_i}$ for τ_{ω_i}, and as in the former Model, simplifications yield

$$
E\left(S_T^2\right) = g_0^{(1)}\left(x\right)^{-1}\sum_{i=0}^{8} f_i^{(1)}\left(x\right)T^i,
$$

where

$$
\begin{aligned}
g_0^{(1)}(x) &\equiv 16800T^2(T-1)(T-3)\left(-2-T^2+4Tx+4T^2x^2\right)^2, \\
f_0^{(1)}(x) &\equiv 768, \\
f_1^{(1)}(x) &\equiv -3072x, \\
f_2^{(1)}(x) &\equiv -4120+8672x^2, \\
f_3^{(1)}(x) &\equiv 8688x+8768x^3, \\
f_4^{(1)}(x) &\equiv -1362+3632x^2-29600x^4, \\
f_5^{(1)}(x) &\equiv 2376x-7936x^3-32896x^5, \\
f_6^{(1)}(x) &\equiv -48+1352x^2+1472x^4+2176x^6, \\
f_7^{(1)}(x) &\equiv -72x-2048x^3+6528x^5+11264x^7, \\
f_8^{(1)}(x) &\equiv 37-72x^2-1024x^4+2176x^6+2816x^8.
\end{aligned}
$$

7.0.4 Model 3

In Model 3, $Z=(1_T,d_1,\tau/T,\tau_{\omega_i}/T)$, and denoting the entries of $(Z'Z)^{-1}$ by $c_{ij}^{(3)}$, we get

$$
\begin{aligned}
& Z(Z'Z)^{-1}Z' \\
= \ & c_{11}^{(3)}1_T1_T'+c_{12}^{(3)}2_c(1_Td_1')+T^{-1}c_{13}^{(3)}2_c(1_T\tau')+T^{-1}c_{14}^{(3)}2_c\left(1_T\tau_{\omega_i}'\right) \\
& +c_{22}^{(3)}d_1d_1'+T^{-1}c_{23}^{(3)}2_c(d_1\tau')+T^{-1}c_{24}^{(3)}2_c\left(d_1\tau_{\omega_i}'\right) \\
& +T^{-2}c_{33}^{(3)}\tau\tau'+T^{-2}c_{34}^{(3)}2_c\left(\tau\tau_{\omega_i}'\right)+T^{-2}c_{44}^{(3)}\tau_{\omega_i}\tau_{\omega_i}'.
\end{aligned}
$$

Hence, we find in the usual manner that

$$
\begin{aligned}
& LZ(Z'Z)^{-1}Z'L' \\
= \ & c_{11}^{(3)}\tau\tau'+c_{12}^{(3)}2_c\left(\tau\tau_{\omega_i}'\right)+T^{-1}c_{13}^{(3)}2_c\left(\tau\xi'\right)+T^{-1}c_{14}^{(3)}2_c\left(\tau\xi_{\omega_i}'\right) \\
& +c_{22}^{(3)}\tau_{\omega_i}\tau_{\omega_i}'+T^{-1}c_{23}^{(3)}2_c\left(\tau_{\omega_i}\xi'\right)+T^{-1}c_{24}^{(3)}2_c\left(\tau_{\omega_i}\xi_{\omega_i}'\right) \\
& +T^{-2}c_{33}^{(3)}\xi\xi'+T^{-2}c_{34}^{(3)}2_c\left(\xi\xi_{\omega_i}'\right)+T^{-2}c_{44}^{(3)}\xi_{\omega_i}\xi_{\omega_i}', \qquad (51)
\end{aligned}
$$

which yields

$$
\begin{aligned}
s_1^* = \ & c_{11}^{(3)}\tau'\tau+2c_{12}^{(3)}\tau_{\omega_i}'\tau+2T^{-1}c_{13}^{(3)}\xi'\tau+2T^{-1}c_{14}^{(3)}\xi_{\omega_i}'\tau \\
& +c_{22}^{(3)}\tau_{\omega_i}'\tau_{\omega_i}+2T^{-1}c_{23}^{(3)}\xi'\tau_{\omega_i}+2T^{-1}c_{24}^{(3)}\xi_{\omega_i}'\tau_{\omega_i} \\
& +T^{-2}c_{33}^{(3)}\xi'\xi+2T^{-2}c_{34}^{(3)}\xi_{\omega_i}'\xi+T^{-2}c_{44}^{(3)}\xi_{\omega_i}'\xi_{\omega_i},
\end{aligned}
$$

where

$$
\xi_{\omega_i}'\tau_{\omega_i}=\frac{1}{2}\sum_{i=1}^{\omega_iT}i^2(i+1)=\frac{1}{24}\omega_iT(\omega_iT+1)(\omega_iT+2)(3\omega_iT+1).
$$

Hence, via (22) (with $p = 4$) and simplifications, we find

$$\begin{aligned} E(S_T) &= T^{-1}(T-4)^{-1}\left(s_1^{(T)} - s_1^*\right) \\ &= \frac{1}{30}T^{-1}(T-4)^{-1}\left(-16 + T^2 + 4T^2x^2\right). \end{aligned}$$

Note the symmetry in x and the surprising (?) resemblance to the corresponding formula in Model 0.

To find the second moment, (51) yields

$$\begin{aligned} & L'LZ(Z'Z)^{-1}Z'L'L \\ = \; & c_{11}^{(3)}\upsilon\upsilon' + c_{12}^{(3)}2_c\left(\upsilon\upsilon'_{\omega_i}\right) + T^{-1}c_{13}^{(3)}2_c\left(\upsilon\phi'\right) + T^{-1}c_{14}^{(3)}2_c\left(\upsilon\phi'_{\omega_i}\right) \\ & + c_{22}^{(3)}\upsilon_{\omega_i}\upsilon'_{\omega_i} + T^{-1}c_{23}^{(3)}2_c\left(\upsilon_{\omega_i}\phi'\right) + T^{-1}c_{24}^{(3)}2_c\left(\upsilon_{\omega_i}\phi'_{\omega_i}\right) \\ & + T^{-2}c_{33}^{(3)}\phi\phi' + T^{-2}c_{34}^{(3)}2_c\left(\phi\phi'_{\omega_i}\right) + T^{-2}c_{44}^{(3)}\phi_{\omega_i}\phi'_{\omega_i}, \end{aligned}$$

and so,

$$\begin{aligned} c_1^{(3)} &\equiv \operatorname{tr}\left\{Z(Z'Z)^{-1}Z'(L'L)^2\right\} \\ &= c_{11}^{(3)}\upsilon'\upsilon + 2c_{12}^{(3)}\upsilon'_{\omega_i}\upsilon + 2T^{-1}c_{13}^{(3)}\phi'\upsilon + 2T^{-1}c_{14}^{(3)}\phi'_{\omega_i}\upsilon \\ &\quad + c_{22}^{(3)}\upsilon'_{\omega_i}\upsilon_{\omega_i} + 2T^{-1}c_{23}^{(3)}\phi'\upsilon_{\omega_i} + 2T^{-1}c_{24}^{(3)}\phi'_{\omega_i}\upsilon_{\omega_i} \\ &\quad + T^{-2}c_{33}^{(3)}\phi'\phi + 2T^{-2}c_{34}^{(3)}\phi'_{\omega_i}\phi + T^{-2}c_{44}^{(3)}\phi'_{\omega_i}\phi_{\omega_i}, \end{aligned}$$

where

$$\begin{aligned} \phi'_{\omega_i}\upsilon_{\omega_i} &= \frac{1}{12}\omega_i^2T^2(\omega_iT+1)^2(\omega_iT+2)(1-\omega_i)T \\ &\quad + \frac{1}{12}\sum_{i=1}^{\omega_iT}(\omega_iT+1-i)^2\left(\omega_i^2T^2 + (2+i)\omega_iT + i + i^2\right)(\omega_iT+i), \\ \phi'\upsilon &= \frac{1}{12}\sum_{i=1}^{T}(T+1-i)^2\left(T^2 + (2+i)T + i + i^2\right)(T+i). \end{aligned}$$

Furthermore, (51) yields

$$\begin{aligned} & \left\{LZ(Z'Z)^{-1}Z'L'\right\}^2 \\ = \; & \Big\{c_{11}^{(3)}\tau\tau' + c_{12}^{(3)}2_c\left(\tau\tau'_{\omega_i}\right) + T^{-1}c_{13}^{(3)}2_c\left(\tau\xi'\right) + T^{-1}c_{14}^{(3)}2_c\left(\tau\xi'_{\omega_i}\right) \\ & + c_{22}^{(3)}\tau_{\omega_i}\tau'_{\omega_i} + T^{-1}c_{23}^{(3)}2_c\left(\tau_{\omega_i}\xi'\right) + T^{-1}c_{24}^{(3)}2_c\left(\tau_{\omega_i}\xi'_{\omega_i}\right) \\ & + T^{-2}c_{33}^{(3)}\xi\xi' + T^{-2}c_{34}^{(3)}2_c\left(\xi\xi'_{\omega_i}\right) + T^{-2}c_{44}^{(3)}\xi_{\omega_i}\xi'_{\omega_i}\Big\}^2, \end{aligned}$$

implying

$$
\begin{aligned}
& c_2^{(3)} \\
\equiv\ & \operatorname{tr}\left[\left\{LZ\left(Z'Z\right)^{-1}Z'L'\right\}^2\right] \\
=\ & c_{11}^{(3)2}\left(\tau'\tau\right)^2+4c_{11}^{(3)}c_{12}^{(3)}\left(\tau_{\omega_i}'\tau\right)\left(\tau'\tau\right)+4T^{-1}c_{11}^{(3)}c_{13}^{(3)}\left(\xi'\tau\right)\left(\tau'\tau\right) \\
& +4T^{-1}c_{11}^{(3)}c_{14}^{(3)}\left(\xi_{\omega_i}'\tau\right)\left(\tau'\tau\right)+2c_{11}^{(3)}c_{22}^{(3)}\left(\tau_{\omega_i}'\tau\right)^2+4T^{-1}c_{11}^{(3)}c_{23}^{(3)}\left(\tau_{\omega_i}'\tau\right)\left(\xi'\tau\right) \\
& +4T^{-1}c_{11}^{(3)}c_{24}^{(3)}\left(\tau_{\omega_i}'\tau\right)\left(\xi_{\omega_i}'\tau\right)+2T^{-2}c_{11}^{(3)}c_{33}^{(3)}\left(\xi'\tau\right)^2+4T^{-2}c_{11}^{(3)}c_{34}^{(3)}\left(\xi'\tau\right)\left(\xi_{\omega_i}'\tau\right) \\
& +2T^{-2}c_{11}^{(3)}c_{44}^{(3)}\left(\xi_{\omega_i}'\tau\right)^2+2c_{12}^{(3)2}\left\{\left(\tau_{\omega_i}'\tau\right)^2+\left(\tau_{\omega_i}'\tau_{\omega_i}\right)\left(\tau'\tau\right)\right\} \\
& +4T^{-1}c_{12}^{(3)}c_{13}^{(3)}\left\{\left(\tau_{\omega_i}'\tau\right)\left(\xi'\tau\right)+\left(\tau'\tau\right)\left(\tau_{\omega_i}'\xi\right)\right\} \\
& +4T^{-1}c_{12}^{(3)}c_{14}^{(3)}\left\{\left(\tau_{\omega_i}'\tau\right)\left(\xi_{\omega_i}'\tau\right)+\left(\tau'\tau\right)\left(\xi_{\omega_i}'\tau_{\omega_i}\right)\right\} \\
& +4c_{12}^{(3)}c_{22}^{(3)}\left(\tau_{\omega_i}'\tau\right)\left(\tau_{\omega_i}'\tau_{\omega_i}\right)+4T^{-1}c_{12}^{(3)}c_{23}^{(3)}\left\{\left(\tau_{\omega_i}'\tau_{\omega_i}\right)\left(\xi'\tau\right)+\left(\tau_{\omega_i}'\tau\right)\left(\tau_{\omega_i}'\xi\right)\right\} \\
& +4T^{-1}c_{12}^{(3)}c_{24}^{(3)}\left\{\left(\tau_{\omega_i}'\tau_{\omega_i}\right)\left(\xi_{\omega_i}'\tau\right)+\left(\tau_{\omega_i}'\tau\right)\left(\xi_{\omega_i}'\tau_{\omega_i}\right)\right\} \\
& +4T^{-2}c_{12}^{(3)}c_{33}^{(3)}\left(\tau_{\omega_i}'\xi\right)\left(\xi'\tau\right)+4T^{-2}c_{12}^{(3)}c_{34}^{(3)}\left\{\left(\xi'\tau\right)\left(\xi_{\omega_i}'\tau_{\omega_i}\right)+\left(\tau_{\omega_i}'\xi\right)\left(\xi_{\omega_i}'\tau\right)\right\} \\
& +4c_{12}^{(3)}c_{44}^{(3)}\left(\xi_{\omega_i}'\tau_{\omega_i}\right)\left(\xi_{\omega_i}'\tau\right) \\
& +2T^{-2}c_{13}^{(3)2}\left\{\left(\xi'\tau\right)^2+\left(\tau'\tau\right)\left(\xi'\xi\right)\right\}+4T^{-2}c_{13}^{(3)}c_{14}^{(3)}\left\{\left(\tau'\tau\right)\left(\xi_{\omega_i}'\xi\right)+\left(\xi_{\omega_i}'\tau\right)\left(\xi'\tau\right)\right\} \\
& +4T^{-1}c_{13}^{(3)}c_{22}^{(3)}\left(\tau_{\omega_i}'\tau\right)\left(\tau_{\omega_i}'\xi\right)+4T^{-2}c_{13}^{(3)}c_{23}^{(3)}\left\{\left(\tau_{\omega_i}'\tau\right)\left(\xi'\xi\right)+\left(\tau_{\omega_i}'\xi\right)\left(\xi'\tau\right)\right\} \\
& +4T^{-2}c_{13}^{(3)}c_{24}^{(3)}\left\{\left(\tau_{\omega_i}'\tau\right)\left(\xi_{\omega_i}'\xi\right)+\left(\tau_{\omega_i}'\xi\right)\left(\xi_{\omega_i}'\tau\right)\right\}+4T^{-3}c_{13}^{(3)}c_{33}^{(3)}\left(\tau'\xi\right)\left(\xi'\xi\right) \\
& +4T^{-3}c_{13}^{(3)}c_{34}^{(3)}\left\{\left(\xi'\xi\right)\left(\xi_{\omega_i}'\tau\right)+\left(\xi'\tau\right)\left(\xi_{\omega_i}'\xi\right)\right\}+4T^{-3}c_{13}^{(3)}c_{44}^{(3)}\left(\xi_{\omega_i}'\xi\right)\left(\xi_{\omega_i}'\tau\right) \\
& +2T^{-2}c_{14}^{(3)2}\left\{\left(\tau'\tau\right)\left(\xi_{\omega_i}'\xi_{\omega_i}\right)+\left(\xi_{\omega_i}'\tau\right)^2\right\}+4T^{-1}c_{14}^{(3)}c_{22}^{(3)}\left(\tau_{\omega_i}'\tau\right)\left(\xi_{\omega_i}'\tau_{\omega_i}\right) \\
& +4T^{-2}c_{14}^{(3)}c_{23}^{(3)}\left\{\left(\tau_{\omega_i}'\tau\right)\left(\xi_{\omega_i}'\xi\right)+\left(\xi'\tau\right)\left(\xi_{\omega_i}'\tau_{\omega_i}\right)\right\} \\
& +4T^{-2}c_{14}^{(3)}c_{24}^{(3)}\left\{\left(\tau_{\omega_i}'\tau\right)\left(\xi_{\omega_i}'\xi_{\omega_i}\right)+\left(\xi_{\omega_i}'\tau\right)\left(\xi_{\omega_i}'\tau_{\omega_i}\right)\right\}+4T^{-3}c_{14}^{(3)}c_{33}^{(3)}\left(\xi'\tau\right)\left(\xi_{\omega_i}'\xi\right) \\
& +4T^{-3}c_{14}^{(3)}c_{34}^{(3)}\left\{\left(\xi'\tau\right)\left(\xi_{\omega_i}'\xi_{\omega_i}\right)+\left(\xi_{\omega_i}'\tau\right)\left(\xi_{\omega_i}'\xi\right)\right\}+4T^{-3}c_{14}^{(3)}c_{44}^{(3)}\left(\xi_{\omega_i}'\tau\right)\left(\xi_{\omega_i}'\xi_{\omega_i}\right) \\
& +c_{22}^{(3)2}\left(\tau_{\omega_i}'\tau_{\omega_i}\right)^2+4T^{-1}c_{22}^{(3)}c_{23}^{(3)}\left(\tau_{\omega_i}'\tau_{\omega_i}\right)\left(\tau_{\omega_i}'\xi\right)+4T^{-1}c_{22}^{(3)}c_{24}^{(3)}\left(\tau_{\omega_i}'\tau_{\omega_i}\right)\left(\xi_{\omega_i}'\tau_{\omega_i}\right) \\
& +2T^{-2}c_{22}^{(3)}c_{33}^{(3)}\left(\tau_{\omega_i}'\xi\right)^2+4T^{-2}c_{22}^{(3)}c_{34}^{(3)}\left(\tau_{\omega_i}'\xi\right)\left(\xi_{\omega_i}'\tau_{\omega_i}\right)+2T^{-2}c_{22}^{(3)}c_{44}^{(3)}\left(\xi_{\omega_i}'\tau_{\omega_i}\right)^2 \\
& +2T^{-2}c_{23}^{(3)2}\left\{\left(\tau_{\omega_i}'\xi\right)^2+\left(\tau_{\omega_i}'\tau_{\omega_i}\right)\left(\xi'\xi\right)\right\} \\
& +4T^{-2}c_{23}^{(3)}c_{24}^{(3)}\left\{\left(\tau_{\omega_i}'\tau_{\omega_i}\right)\left(\xi_{\omega_i}'\xi\right)+\left(\tau_{\omega_i}'\xi\right)\left(\xi_{\omega_i}'\tau_{\omega_i}\right)\right\}+4T^{-3}c_{23}^{(3)}c_{33}^{(3)}\left(\tau_{\omega_i}'\xi\right)\left(\xi'\xi\right) \\
& +4T^{-3}c_{23}^{(3)}c_{34}^{(3)}\left\{\left(\xi_{\omega_i}'\xi\right)\left(\tau_{\omega_i}'\xi\right)+\left(\xi_{\omega_i}'\tau_{\omega_i}\right)\left(\xi'\xi\right)\right\}+4T^{-3}c_{23}^{(3)}c_{44}^{(3)}\left(\xi_{\omega_i}'\xi\right)\left(\xi_{\omega_i}'\tau_{\omega_i}\right) \\
& +2T^{-2}c_{24}^{(3)2}\left\{\left(\xi_{\omega_i}'\tau_{\omega_i}\right)^2+\left(\tau_{\omega_i}'\tau_{\omega_i}\right)\left(\xi_{\omega_i}'\xi_{\omega_i}\right)\right\}+4T^{-3}c_{24}^{(3)}c_{33}^{(3)}\left(\xi_{\omega_i}'\xi\right)\left(\tau_{\omega_i}'\xi\right) \\
& +4T^{-3}c_{24}^{(3)}c_{34}^{(3)}\left\{\left(\xi_{\omega_i}'\xi_{\omega_i}\right)\left(\tau_{\omega_i}'\xi\right)+\left(\xi_{\omega_i}'\tau_{\omega_i}\right)\left(\xi_{\omega_i}'\xi\right)\right\}+4T^{-3}c_{24}^{(3)}c_{44}^{(3)}\left(\xi_{\omega_i}'\xi_{\omega_i}\right)\left(\xi_{\omega_i}'\tau_{\omega_i}\right) \\
& +T^{-4}c_{33}^{(3)2}\left(\xi'\xi\right)^2+4T^{-4}c_{33}^{(3)}c_{34}^{(3)}\left(\xi_{\omega_i}'\xi\right)\left(\xi'\xi\right)+2T^{-4}c_{33}^{(3)}c_{44}^{(3)}\left(\xi_{\omega_i}'\xi\right)^2 \\
& +2T^{-4}c_{34}^{(3)2}\left\{\left(\xi_{\omega_i}'\xi\right)^2+\left(\xi_{\omega_i}'\xi_{\omega_i}\right)\left(\xi'\xi\right)\right\}+4T^{-4}c_{34}^{(3)}c_{44}^{(3)}\left(\xi_{\omega_i}'\xi_{\omega_i}\right)\left(\xi_{\omega_i}'\xi\right) \\
& +T^{-4}c_{44}^{(3)2}\left(\xi_{\omega_i}'\xi_{\omega_i}\right)^2.
\end{aligned}
$$

In the usual way, this yields

$$\begin{aligned} & E\left(S_T^2\right) \\ = \;& T^{-2}\left(T-2\right)^{-1}\left(T-4\right)^{-1}\left\{\left(s_1^{(T)}-s_1^*\right)^2+2(b-s_2^*)\right\} \\ = \;& \left\{50400(T-4)(T-2)T^2\right\}^{-1} \\ & \left(2752-1244T^2+67T^4-4976T^2x^2+712T^4x^2+1072T^4x^4\right). \end{aligned}$$

Proof of Theorem 1 using Laplace Transform

As in Hadri and Larsson (2002), we get

$$E\left(S_T\right) = T^{-1}\int_0^\infty \det P_0^{-1/2} a_1 dr, \tag{52}$$

$$E\left(S_T^2\right) = T^{-2}\int_0^\infty \det P_0^{-1/2} a_2 r dr, \tag{53}$$

where

$$P_0 \equiv I_T + 2rM^2, \tag{54}$$

$$a_1 \equiv \operatorname{tr}\left(P_0^{-1}M'L'LM\right), \tag{55}$$

$$a_2 \equiv \left\{\operatorname{tr}\left(P_0^{-1}M'L'LM\right)\right\}^2 + 2\operatorname{tr}\left\{\left(P_0^{-1}M'L'LM\right)^2\right\}. \tag{56}$$

To find $\det P_0$, observe that

$$M^2 = M = I_T - Z(Z'Z)^{-1}Z', \tag{57}$$

implying

$$P_0 = (1+2r)\left\{I_T - \frac{2r}{1+2r}Z(Z'Z)^{-1}Z'\right\},$$

and the identity

$$\det\left(I_n - VW\right) = \det\left(I_m - WV\right),$$

where V and W are arbitrary $n\times m$ and $m\times n$ matrices, respectively, yields, for a $T\times p$ Z matrix,

$$\det P_0 = (1+2r)^T \det\left(I_p - \frac{2r}{1+2r}I_p\right) = (1+2r)^{T-p}. \tag{58}$$

Moreover, it is easily seen that

$$P_0^{-1} = (1+2r)^{-1}\left\{I_T + 2rZ(Z'Z)^{-1}Z'\right\}, \tag{59}$$

and so, (55) yields

$$\begin{aligned} a_1 &= (1+2r)^{-1}\left[\operatorname{tr}\left(M'L'LM\right) + 2r\operatorname{tr}\left\{Z(Z'Z)^{-1}Z'M'L'LM\right\}\right] \\ &= (1+2r)^{-1}\operatorname{tr}\left(M'L'LM\right), \end{aligned} \tag{60}$$

where the second equality follows from $MZ = 0$. Hence, because L and M do not depend on r, we get via (58) and (52) that

$$\begin{aligned} E(S_T) &= T^{-1}\int_0^\infty (1+2r)^{-(T-p)/2-1}\,dr\,\mathrm{tr}\,(M'L'LM) \\ &= T^{-1}(T-p)^{-1}\,\mathrm{tr}\,(M'L'LM). \end{aligned} \tag{61}$$

Further, via (57),

$$\begin{aligned} \mathrm{tr}\,(M'L'LM) &= \mathrm{tr}\left(M^2L'L\right) = \mathrm{tr}\,(ML'L) = \mathrm{tr}\left\{L'L - Z(Z'Z)^{-1}Z'L'L\right\} \\ &= \mathrm{tr}\,(L'L) - \mathrm{tr}\left\{LZ(Z'Z)^{-1}Z'L'\right\}. \end{aligned} \tag{62}$$

Here, from (14),

$$\mathrm{tr}\,(L'L) = s_1^{(T)}. \tag{63}$$

Hence, (61) yields

$$E(S_T) = T^{-1}(T-p)^{-1}\left(s_1^{(T)} - s_1^*\right), \quad s_1^* \equiv \mathrm{tr}\left\{LZ(Z'Z)^{-1}Z'L'\right\}. \tag{64}$$

The value of s_1^* depends on the specification of Z. To find the second moment, we at first use (59) and $MZ = 0$ to find

$$MP_0^{-1} = (1+2r)^{-1}M,$$

implying

$$\begin{aligned} &\mathrm{tr}\left\{\left(P_0^{-1}M'L'LM\right)^2\right\} \\ = \;&\mathrm{tr}\left\{\left(MP_0^{-1}M'L'L\right)^2\right\} = (1+2r)^{-2}\,\mathrm{tr}\left\{(MM'L'L)^2\right\} \\ = \;&(1+2r)^{-2}\,\mathrm{tr}\left\{(ML'L)^2\right\}. \end{aligned} \tag{65}$$

Moreover,

$$\begin{aligned} &\mathrm{tr}\left\{(ML'L)^2\right\} \\ = \;&\mathrm{tr}\left[\left\{\left(I_T - Z(Z'Z)^{-1}Z'\right)L'L\right\}^2\right] \\ = \;&\mathrm{tr}\left\{(L'L)^2\right\} - 2\,\mathrm{tr}\left\{Z(Z'Z)^{-1}Z'\,(L'L)^2\right\} + \mathrm{tr}\left[\left\{LZ(Z'Z)^{-1}Z'L'\right\}^2\right]. \end{aligned}$$

Hence, because

$$\begin{aligned} \mathrm{tr}\left\{(L'L)^2\right\} &= 1^2(2T-1) + 2^2(2T-3) + ... + T^2\cdot 1 \\ &= \frac{1}{6}T(T+1)\left(T^2+T+1\right) \equiv b, \end{aligned} \tag{66}$$

(cf Hadri and Larsson (2002)), we have

$$\mathrm{tr}\left\{(ML'L)^2\right\} = b - s_2^*,$$

where

$$s_2^* \equiv 2\,\mathrm{tr}\left\{Z(Z'Z)^{-1}Z'\left(L'L\right)^2\right\} - \mathrm{tr}\left[\left\{LZ(Z'Z)^{-1}Z'L'\right\}^2\right]. \tag{67}$$

Thus, via (65) inserting together with (62)-(22) and (60) into (56), we get

$$\begin{aligned} a_2 &= (1+2r)^{-2}\left\{\mathrm{tr}\left(M'L'LM\right)\right\}^2 + 2\,(1+2r)^{-2}\,\mathrm{tr}\left\{\left(ML'L\right)^2\right\} \\ &= (1+2r)^{-2}\left(s_1^{(T)} - s_1^*\right)^2 + 2\,(1+2r)^{-2}\left(b - s_2^*\right), \end{aligned}$$

and so, using (53) and (58),

$$\begin{aligned} & E\left(S_T^2\right) \qquad (68) \\ = \;& T^{-2}\int_0^\infty (1+2r)^{-(T-p)/2-2}\,a_2 r dr \\ = \;& T^{-2}\int_0^\infty (1+2r)^{-(T-p)/2-2}\,r dr\left(s_1^{(T)} - s_1^*\right)^2 \\ & \left\{\left(s_1^{(T)} - s_1^*\right)^2 + 2\left(b - s_2^*\right)\right\} \\ = \;& T^{-2}\left(T-p+2\right)^{-1}\left(T-p\right)^{-1}\left\{\left(s_1^{(T)} - s_1^*\right)^2 + 2\left(b - s_2^*\right)\right\}. \qquad (69) \end{aligned}$$

The values of s_1^* and s_2^* depend on the choice of Z. The rest of the proof is the same as above.